Part VIII

第八部分

Causal Dynamical Triangulations

因果动力学三角剖分

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Lattice Quantum Gravity: EDT and CDT

格点量子引力: 欧几里得动力学三角剖分与因果动力学三角剖分

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Contents

目录

Abstract

摘要

This chapter contains an overview of the use of so-called Euclidean dynamical triangulations (EDT) and causal dynamical triangulations (CDT) as lattice regularizations of quantum gravity. The lattice regularizations have been very successful in the case of two-dimensional quantum gravity, where the lattice theories indeed provide regularizations of continuum well-defined quantum gravity theories. In four-dimensional spacetime the Einstein-Hilbert action leads to a theory of gravity which is not renormalizable as a perturbative quantum theory around flat spacetime. It is discussed how lattice gravity in the form of EDT or CDT can be used to search for a non-perturbative UV fixed point of the lattice renormalization group in the spirit of asymptotic safety. In this way it might be possible to define a quantum theory of gravity also at length scales smaller than the Planck length.

本章概述欧几里得动力三角剖分 (EDT) 与因果动力三角剖分 (CDT) 作为量子引力的格点正则化方法的应用。该格点正则化在二维量子引力领域已取得极大成功，在此情形下，格点理论确实为连续统中定义良好的量子引力理论提供了正则化。在四维时空中，爱因斯坦-希尔伯特作用量导出的引力理论作为平直时空背景下的微扰量子理论是不可重整的。本文讨论了如何利用 EDT 或 CDT 形式的格点引力，按照渐近安全的思想寻找格点重整化群的非微扰紫外不动点。通过这种方式，或许能够在小于普朗克长度的尺度上也定义出量子引力理论。

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量子引力。格引力——欧几里得动力三角化。因果动力三角化——量子几何——量子引力

Introduction

引言

So far there is no universally accepted quantum theory of four-dimensional gravity. The classical theory of general relativity is not perturbative renormalizable. Therefore, if we think about four-dimensional quantum gravity as small quantum fluctuations around some classical geometry that solves Einstein's equations, it only makes sense as an effective quantum field theory up to some energy or down to some length scale, determined by the coupling constants entering in the classical theory (this is discussed in detail in the section

”Effective Quantum Gravity” in the handbook). Unfortunately, we have presently no experiments which can guide us if we want to approach or go beyond this scale, which is the Planck energy or the Planck length (If G denotes Newton’s gravitational constant, c the velocity of light, and $\hbar$ the Planck constant, the Planck energy is $E_p = \sqrt{\hbar c^5/G}$ and the Planck length is $\ell_p = \sqrt{\hbar G/c^3}$.). There are examples of other quantum field theories which, when viewed at low energies, appeared to be non-renormalizable, like the theory of weak interactions and the theory of strong interactions. In the case of the weak interactions, the four-fermion interaction, originally suggested to explain the weak interactions, is non-renormalizable. However, we now know that at high energies it is resolved into renormalizable interactions mediated by W and Z particles. Similarly, the non-renormalizable nonlinear sigma model, which was used to describe the low-energy $\pi - \pi$ interaction related to the strong interactions, is a low-energy effective action of an underlying renormalizable quantum field theory of quarks and gluons. From these examples it is tempting to conjecture that the same could happen to quantum gravity and that the non-renormalizability of gravity would be resolved at larger energy by new degrees of freedom that we have not yet observed. It could be the case, but gravity still looks different from these two examples. In the case of the weak interactions one was led to the four-fermion interaction and not to the renormalizable version of the weak interactions simply because the W and Z particles were so heavy that they had not yet been observed. In the case of the strong interactions one was led to a $\pi - \pi$ interaction because the quark and gluons were not observed, not because they were heavy, but because of quark and gluon confinement. In both cases the starting points were really (effective) quantum theories, the classical aspects of the theories playing minor roles. In gravity the situation is different. We have a classical theory, which seemingly works very well, and this theory even has long-range massless classical excitations propagating with the velocity of light in a classical background geometry, the now famous gravitational waves. It does not seem too promising to try to explain this as a limit of a renormalizable quantum theory constructed from heavy, yet to be observed, fundamental particles, or from ”confined” light particles.

迄今为止，尚未存在被普遍接受的四维引力量子理论。经典广义相对论不满足微扰可重整化。因此，如果我们将四维量子引力视为爱因斯坦方程经典解背景下的小量子涨落，那么它仅能作为有效量子场论成立，适用范围是高于某一能量标度或低于某一长度标度，该标度由经典理论中的耦合常数决定 (本手册“有效量子引力”章节对此有详细讨论)。遗憾的是，目前我们没有实验能指导我们研究达到或超越该标度——即普朗克能量或普朗克长度 (若 G 表示牛顿引力常数，c 表示光速，$\hbar$ 表示普朗克常数，则普朗克能量为 $E_p = \sqrt{\hbar c^5/G}$，普朗克长度为 $\ell_p = \sqrt{\hbar G/c^3}$)。其他量子场论中也有低能下看似不可重整化的例子，例如弱相互作用理论和强相互作用理论。最初为解释弱相互作用提出的四费米子相互作用就是不可重整化的，但我们现在知道，它在高能下可分解为由 W 和 Z 粒子传递的可重整化相互作用。类似地，用于描述强相互作用相关低能 $\pi - \pi$ 相互作用的不可重整化非线性 σ 模型，其实是夸克胶子底层可重整化量子场论的低能有效作用量。从这些例子出发，人们很容易猜想量子引力也会如此: 引力的不可重整化问题会在更高能标下被我们尚未观测到的新自由度解决。这种情况确实有可能成立，但引力和这两个例子仍有区别。弱相互作用研究中，人们最初得到四费米子相互作用而非可重整化形式，仅仅是因为 W 和 Z 粒子质量太大，当时尚未被观测到; 强相互作用研究中，人们得到 $\pi - \pi$ 相互作用是因为夸克和胶子未被观测到，这并非因为它们质量大，而是源于夸克胶子禁闭。两种情况的起点本质上都是 (有效) 量子理论，经典性质仅起次要作用。引力的情况则不同: 我们拥有一个看似非常成功的经典理论，该理论甚至存在长程无质量经典激发，即以光速在经典背景几何中传播的著名引力波。想要将引力解释为由尚未观测到的重基本粒子，或是“被禁闭”的轻粒子所构建的可重整化量子理论的极限，似乎希望不大。

String theory is one attempt to provide a quantum theory of gravity. More precisely, closed string theory contains massless spin-two excitations, which can be interpreted as quantum gravity particles and the underlying stringy nature of the theory solves the UV problems associated with quantizing the Einstein-Hilbert action of classical gravity. The original hope was that the (super)string theory would provide us with an explanation of all the particles we actually observe in nature, and at the same time it predicted the existence of particles we have not yet observed. Unfortunately, no clear picture related to the world we observe has yet emerged from string theory, which, since the ambitious start in the 1980s as a theory of everything, has developed in many directions. The directions still related to gravity will be described in the section of the handbook dedicated to string theory.

弦理论是构建引力量子理论的一种尝试。更准确地说，闭弦理论包含无质量自旋二激发，可被解释为量子引力粒子，该理论底层的弦本质解决了经典引力爱因斯坦-希尔伯特作用量量子化带来的紫外问题。最初的期望是 (超) 弦理论能解释我们在自然界实际观测到的所有粒子，同时它也预言了尚未观测到的粒子的存在。遗憾的是，弦理论自 20 世纪 80 年代作为“万有理论”雄心勃勃起步后，已经向诸多方向发展，至今仍未呈现出和我们观测到的世界相关的清晰图景。本手册中专门讨论弦理论的章节会介绍仍与引力相关的研究方向。

Loop quantum gravity is another attempt to circumvent the problem associated with a ”naive” quantization of gravity based on the Einstein-Hilbert action. It deals with the UV problems of the naive approach by postulating a new quantization procedure which leads to a Hilbert space quite different from the standard Fock space of particle physics. This procedure defines in principle the physics at the Planck scale, but it becomes difficult to relate the theory to classical gravity as we observe it today. Like string theory it has branched out in a number of different directions, again described in the handbook in the section about loop quantum gravity.

圈量子引力是另一种尝试，旨在解决基于爱因斯坦-希尔伯特作用量对引力进行“朴素”量子化带来的问题。它通过引入新的量子化程序处理朴素方法的紫外问题，该程序得到的希尔伯特空间与粒子物理中标准的福克空间完全不同。该程序原则上可以定义普朗克尺度下的物理，但很难将该理论和我们如今观测到的经典引力联系起来。和弦理论类似，圈量子引力也分化出诸多不同方向，本手册中关于圈量子引力的章节会对此进行介绍。

Lattice quantum gravity, as described in this section, is closely related to a field theoretical approach to quantum gravity using what is called asymptotic safety. A special section in the handbook is dedicated to the use of asymptotic safety when quantizing gravity. The hypothesis is that one can use ordinary quantum field theory to quantize gravity and that the UV limit of the theory corresponds to a non-perturbative fixed point. Around this fixed point one cannot apply ordinary perturbation theory, by expanding in a power series of coupling constants appearing in the classical low-energy Lagrangian. Nevertheless, it is postulated that there exists a continuum renormalization group flow of the effective action that will reach the UV fixed point by adjusting only a finite number of suitable coupling constants. In this sense the non-perturbative fixed point is similar to a (Gaussian) UV fixed point of a renormalizable quantum field theory. It is in this context that lattice quantum gravity becomes interesting for a number of reasons.

本节所述的格点量子引力与采用渐近安全思想的量子引力场论方法密切相关。本手册已有专门章节介绍量子化引力时渐近安全的应用。该假说认为，我们可以使用普通量子场论对引力进行量子化，该理论的紫外极限对应一个非微扰不动点。在该不动点周围，无法通过对经典低能拉格朗日量中的耦合常数做幂级数展开来应用普通微扰论。尽管如此，该假说假定，有效作用量存在连续重整化群流，仅需调整有限个合适的耦合常数，就能使该流到达紫外不动点。从这个意义上说，这个非微扰不动点与可重整量子场论的 (高斯) 紫外不动点类似。正是在这一背景下，格点量子引力因诸多原因变得颇具研究价值。

From a Wilsonian point of view lattice field theory is well suited to studying fixed points and renormalization group flows, as well as non-perturbative aspects of the corresponding quantum field theories. The lattice will provide a UV regularization of the quantum field theory in question. Such a UV regularization is usually needed as starting point for defining an interacting quantum field theory. In order to define the corresponding continuum quantum field theory, one will in general need to take the UV cutoff, i.e., the lattice spacing, to zero relative to some continuum length scale characterizing the continuum theory. If the theory contains massive particles, one can use the inverse mass of such a particle as the length scale (in units where $c = \hbar = 1$). Keeping such a physical length scale fixed while taking the lattice spacing to zero implies that this length scale measured in lattice units will diverge. This is most simply realized in lattice field theories if a correlator of one of the lattice fields for a generic choice of the lattice coupling constants g_i of the theory is decaying exponentially with the distance between the lattice points, in this way defining a correlation length $\xi(g_i)$. A second- (or higher-) order phase transition of the lattice field theory is often characterized by a divergent correlation length. Thus, when trying to find continuum limits of the lattice field theory, it is natural to look for regions in the lattice coupling space associated with second- or higher-order phase transitions. One of the assumptions in the Wilsonian approach, if one considers lattice field Hamiltonians with arbitrary local interactions, i.e., in principle an infinite-dimensional coupling constant space, is that the critical surface of higher-order phase transitions has a finite co-dimension. In this case one only has to fine-tune a finite number of coupling constants to reach the critical surface where the lattice correlation length is infinite. The way in which one approaches the critical surface will define the continuum physical parameters of the corresponding continuum quantum field theory (like the masses of the particles corresponding to lattice field correlators), and the nature and the number of lattice coupling constants which need to be fine-tuned to reach the critical surface will depend on the so-called fixed points of the lattice renormalization group. These fixed points are located on the critical surface, and in the Wilsonian picture each can be used to define a continuum quantum field theory.

从威尔森的视角来看，格点场论非常适合研究不动点、重整化群流，以及对应量子场论的非微扰性质。格点会为目标量子场论提供紫外正则化。这种紫外正则化通常是定义相互作用量子场论所需的起点。要定义对应的连续量子场论，一般需要将紫外截断 (即晶格间距)，相对于表征连续理论的某一连续长度尺度取零。若理论包含有质量粒子，可以将该粒子的质量倒数用作长度尺度 (单位制满足 $c = \hbar = 1$)。在格点间距取零的同时保持该物理长度尺度固定，意味着以格点单位测量得到的该长度尺度会发散。对于格点场论而言，当理论的格点耦合常数 g_i 取任意一般选择时，若某个格点场的关联函数随格点间距呈指数衰减，以此定义关联长度 $\xi(g_i)$ ，就是实现上述情况最简单的方式。格点场论的二级 (或更高级别) 相变通常以关联长度发散为特征。因此，在寻找格点场论的连续极限时，自然会去格点耦合空间中搜寻与二级或更高级别相变相关的区域。威尔森方法在考虑包含任意局域相互作用的格点场哈密顿量 (原则上是无穷维耦合常数空间) 时的一个假设是，高级别相变的临界面具有有限余维数。这种情况下，仅需对有限个耦合常数做微调，就能到达格点关联长度为无穷大的临界面。趋近临界面的方式会定义出对应连续量子场论的连续物理参数 (比如对应格点场关联子的粒子质量)，而要到达临界面所需微调的格点耦合常数的性质与数量，取决于格点重整化群的所谓不动点。这些不动点位于临界面上，在威尔森的图景中，每个不动点都可用于定义一个连续量子场论。

We want to use this lattice Wilsonian framework to investigate whether we can define a continuum limit of theories we can denote "lattice gravity" theories, more precisely, the lattice gravity theories based on Euclidean dynamical triangulations (EDT) or causal dynamical triangulations (CDT). We have a space of (dimensionless) lattice coupling constants associated with the theory and want to locate regions in this space where there are second- (or higher-) order phase transitions. When such regions are localized, we want to understand whether one can approach these phase transition regions such that one obtains a theory that can be viewed as the quantum theory of gravity. It is of particular interest if the phase transition surface can be associated with a UV fixed point, since in this case one might have defined the quantum gravity theory at arbitrarily short distances.

我们希望利用这种格点威尔森框架，探究能否定义我们称之为“格点引力”理论的连续极限，更具体地说，就是基于欧几里得动力学三角剖分 (EDT) 或因果动力学三角剖分 (CDT) 的格点引力理论。我们拥有该理论对应的 (无量纲) 格点耦合常数空间，希望在该空间中定位存在二级 (或更高级别) 相变的区域。定位到这类区域后，我们希望弄清楚能否趋近这些相变区域，从而得到一个可被视为量子引力理论的理论。如果相变面可以和一个紫外不动点对应起来，研究会尤其有意义，因为在这种情况下，我们有可能在任意短距离尺度上定义量子引力理论。

A number of interesting conceptual problems are associated with quantum gravity and the Wilsonian lattice renormalization group. The central Wilsonian idea is that a divergent lattice correlation length of some observable makes it possible to forget the underlying lattice and that using a limiting procedure makes it possible to define a continuum quantum field theory. It is also the reason for the universality associated with the Wilsonian approach: the details of the local lattice structure as well as the details of the interactions at lattice distances are often of no consequence for the continuum limit. Global symmetries of the interactions might be (and are) important as they can survive when a continuum limit is taken. However, when trying to apply this line of reasoning to a lattice gravity theory, we are faced with the very simple question: how does one define a correlation length in a theory of quantum gravity? When implementing the quantum theory via a path integral, we are instructed to integrate over all geometries, but it is the geometries which define the distances. In nongravitational relativistic quantum field theory, correlators are functions of spacetime points, and the main reason we study these correlators is that their behavior as a function of the distances between these spacetime points tells us a lot about the underlying quantum theory. They are also the natural objects on which the renormalization group acts. Thus, it is somewhat disturbing that it is unclear how to define such correlators in a theory of quantum gravity in a way that relates to distances. A common, and in general healthy, attitude in theoretical physics is to calculate whatever can be calculated and postpone annoying questions like how to define distances in a theory of quantum gravity. However, one nice thing about a lattice theory of gravity is that one is forced to address such questions. Four-dimensional lattice gravity cannot in any way be solved analytically, but one can perform computer simulations of the lattice theory. If one wants to measure anything but the simplest global observables in such computer simulations, one should better have a precise idea of how to define the observable to measure in a sensible way. In the rest of this chapter, we will discuss lattice quantum gravity from this Wilsonian point of view and how one can in principle use the lattice approach to test the asymptotic safety conjecture.

量子引力与威尔逊格点重整化群存在诸多有趣的概念问题。威尔逊的核心思想是: 若某可观测量的格点关联长度发散，我们就无需再考虑基础格点结构，通过极限过程即可定义连续量子场论。这也是威尔逊方法具备普适性的原因: 局域格点结构的细节，以及格点间距下相互作用的细节，通常对连续极限没有影响。相互作用的整体对称性可能 (并且确实) 十分重要，因为它们在取连续极限时能够保留下来。但当我们尝试将这套逻辑应用于格点引力理论时，就会面临一个十分直白的问题: 该如何在量子引力理论中定义关联长度? 当我们通过路径积分实现量子理论时，需要对所有几何积分，而正是几何本身定义了距离。在非引力相对论量子场论中，关联函数是时空点的函数，我们研究这些关联函数的核心原因是，它们随时空点间距变化的行为能够为我们揭示基础量子理论的大量信息，关联函数也是重整化群作用的自然对象。因此，如何在量子引力理论中按照和距离关联的方式定义这类关联函数至今尚不明确，这一点相当令人不安。理论物理学中有一种常见且大体而言无害的处理思路: 先计算所有能计算的量，把“如何在量子引力中定义距离”这类棘手问题往后搁置。但格点引力理论的一个特点在于，它迫使我们直面这类问题。四维格点引力完全无法解析求解，但我们可以对该格点理论开展计算机模拟。如果想要在这类计算机模拟中测量除最简单整体可观测量之外的物理量，就必须先明确该如何合理定义要测量的可观测量。在本章剩余部分，我们将从威尔逊的视角讨论格点量子引力，以及原则上如何利用格点方法检验渐近安全猜想。

A Wilsonian View on Two-Dimensional EDT and CDT

二维 EDT 和 CDT 的威尔逊观点

The EDT Partition Functions

EDT 配分函数

Two-dimensional gravity is classically a trivial theory since there are no propagating gravitons in two-dimensional spacetime. One reflection of this is that the curvature term in the Einstein-Hilbert action is a topological invariant in two dimensions. Thus, as long as one does not consider topology changes, and we will not do that, the action just contains the cosmological term, a term with no derivatives of the metric. If we consider spacetimes with Euclidean signature, we have the two-dimensional partition function

二维引力在经典层面是一个平凡理论，因为二维时空中不存在传播引力子。这一点的一个体现是，爱因斯坦-希尔伯特作用量中的曲率项在二维空间中是拓扑不变量。因此，只要我们不考虑拓扑变化——我们在这里也确实不考虑——作用量就仅包含宇宙学项，这一项不含度规的导数项。如果我们考虑欧几里得符号的时空，我们就能得到二维配分函数

$$Z(G,\Lambda,Z_i)=\int\mathcal{D}[g]\,\mathrm{e}^{-S[g]}, \tag{1}$$

$$S[g]=-\frac{1}{2\pi G}\int d^2\xi\sqrt{g}\left(R(\xi)-2\widetilde{\Lambda}\right)=-\frac{\chi}{G}+\Lambda V[g]+\sum_{i=1}^{n}Z_iL_i[g]. \tag{2}$$

The path integral (1) is over all geometries $[g_{ab}]$ (i.e., metrics g_{ab} up to diffeomorphism equivalence) on a two-dimensional manifold with h handles and n boundaries and with Euler characteristic $\chi=2-2h-n.\Lambda$ is the cosmological constant (divided by πG) and Z_i are suitable boundary cosmological constants, which are only introduced for later convenience. $V[g]$ denotes the two-dimensional volume of the manifold, while $L_i[g]$ denotes the length of the i th boundary, all measured in the geometry $[g_{ab}]$. In the following we will ignore the topological term and only consider manifolds with the topology of a sphere with boundaries. The partition function (1) can be written as

路径积分 (1) 遍及具有 h 个柄、n 个边界，欧拉示性数为 $\chi=2-2h-n.\Lambda$ 的二维流形上的所有几何 $[g_{ab}]$ (即对微分同胚等价意义下的度规 g_{ab})，Λ 是宇宙学常数 (除以 πG)，Z_i 是合适的边界宇宙学常数，引入它们只是为了后续推导方便。$V[g]$ 表示流形的二维体积，$L_i[g]$ 表示第 i 个边界的长度，所有量都在几何 $[g_{ab}]$ 下测量。下文我们将忽略拓扑项，仅考虑带边界的球面拓扑的流形。配分函数 (1) 可以写为

$$Z(\Lambda,Z_i)=\int_0^\infty dV\int_0^\infty\prod_{i=1}^{n}dL_i\mathrm{e}^{-\Lambda V-L_iZ_i}\int\mathcal{D}_{V,L_i}[g] \tag{3}$$

$$=\int_0^\infty dV\int_0^\infty\prod_{i=1}^{n}dL_i\mathrm{e}^{-\Lambda V-L_iZ_i}\mathcal{N}(V,L_i). \tag{4}$$

[g] denotes a geometry and in (3) the functional integration is over all geometries which have spacetime volume V and n boundaries of lengths L_i . This integration is formally equal to the number of such geometries. In other words, we can compute the partition function of two-dimensional quantum gravity if we can count the number of geometries with a given spacetime volume and given lengths of the boundaries. Moreover, from this perspective the partition function $Z(\Lambda,Z_i)$ can be viewed as the generating function for the numbers $\mathcal{N}(V,L_i)$, with $e^{-\Lambda}$ and e^{-Z_i} playing the role of indeterminates in this generating function. Of course $\mathcal{N}(V,L_i)$ is formally infinite, reflecting the fact that the path integral $\int\mathcal{D}_{V,L_i}[g]$ needs a UV cutoff to be defined in the first place. The so-called (Euclidean) dynamical triangulations (EDT) provide a useful regularization (Historically, the main interest in the EDT regularization was linked to the use as a regularization of the Polyakov path integral for the bosonic string in D -dimensional spacetime [12-14,44,45,50,57]. This path integral can be viewed as two-dimensional quantum gravity coupled to D bosonic fields X_i , constituting the D coordinates of the bosonic string. Unfortunately, the approach did not work when implemented in the simplest way for $D>1$ as shown in [2]. However, for $D<1$ it has been very successful and known as ”noncritical” string theory, as will be mentioned below. Pure two-dimensional quantum gravity, which we discuss here, corresponds in this context to $D=0$ and was first introduced in [48] and discussed in [57].

There are interesting indications that the formalism can be revived as a regularization of bosonic strings for $D > 1$ by taking a new kind of scaling limit [8-10].). In its simplest version one approximates the integration over geometries by a summation over triangulations constructed from equilateral triangles with link length a , where a serves as a UV cutoff. To each such triangulation one can associate a piecewise linear geometry by assuming the triangles are flat in the interior. The curvature of the piecewise linear geometry is then naturally located at the vertices. Summing over such triangulations one obtains an approximation to the continuum partition function (1) when one makes the identification

[g] 表示一个几何，(3) 中的泛函积分遍及所有时空体积为 V 、边界长度为 L_i 的 n 个边界的几何。这个积分形式上等于这类几何的数量。换言之，如果我们能够计数给定时空体积和给定边界长度的几何数量，我们就能计算二维量子引力的配分函数。此外，从这个角度看，配分函数 $Z(\Lambda, Z_i)$ 可以看作是数量 $\mathcal{N}(V, L_i)$ 的生成函数，其中 $e^{-\Lambda}$ 和 e^{-Z_i} 在这个生成函数中扮演不定元的角色。当然 $\mathcal{N}(V, L_i)$ 形式上是无穷大，这反映了路径积分 $\int \mathcal{D}_{V,L_i}$ [g] 首先就需要紫外截断才能定义。所谓 (欧几里得) 动态三角化 (EDT) 提供了一个有用的正则化 (历史上，对 EDT 正则化的主要兴趣与将它用作 D 维时空中玻色弦波利科夫路径积分的正则化有关 [12-14,44,45,50,57]。该路径积分可以看作是二维量子引力与 D 个玻色场 X_i 耦合，构成了玻色弦的 D 个坐标。遗憾的是，如文献 [2] 所示，当该方法以最简单的形式应用于 $D > 1$ 时并不成立。但对于 $D < 1$ ，它非常成功，被称为“非临界”弦理论，下文将会提及。我们在这里讨论的纯二维量子引力在该框架下对应 $D = 0$ ，它最早在文献 [48] 中提出，在文献 [57] 中讨论。有研究表明，通过取一种新型标度极限，该形式体系可以重新作为 $D > 1$ 下玻色弦的正则化 [8-10]。)。在其最简版本中，我们通过对由链接长度为 a 的等边三角形构造出的三角剖分求和，来近似对几何的积分，其中 a 充当紫外截断。假设三角形内部是平坦的，我们就能为每个这样的三角剖分关联一个分段线性几何，分段线性几何的曲率自然位于顶点处。当我们做出如下识别后，对这类三角剖分求和就可以得到连续配分函数 (1) 的近似

$$V(T) = \frac{\sqrt{3}}{2} N(T)\, a^2, \; L_i(T) = l_i(T)\, a, \tag{5}$$

where $N(T)$ is the number of triangles and $l_i(T)$ the number of boundary links of the i th boundary in the triangulation T . The lattice gravity partition function can be written as (It is assumed that a link on each boundary is marked, in order to avoid symmetry factors appearing in the sum over triangulations.)

其中 $N(T)$ 是三角形的数量，$l_i(T)$ 是三角剖分 T 中第 i 个边界的边界链接数。格点引力配分函数可写为 (为避免三角剖分求和中出现对称因子，假设每个边界上都有一个链接被标记)。

$$Z(\mu, \lambda_i) = \sum_T \mathrm{e}^{-\mu N(T) - \sum_i \lambda_i l_i(T)} = \sum_{N, l_i} \mathrm{e}^{-\mu N - \sum_i \lambda_i l_i} \mathcal{N}(N, l_i), \tag{6}$$

which is the lattice version of (3) and (4), the integration over geometries being replaced by the summation over equilateral triangles with link length a , and with

这是式 (3) 和 (4) 的格点版本，几何积分被替换为对链接长度为 a 的等边三角形求和，且有

$$\mu = \Lambda_0 a^2, \; \lambda_i = Z_i^{(0)} a. \tag{7}$$

We call Λ_0 and $Z_i^{(0)}$ the bare, unrenormalized coupling constants for reasons that will become clear below. By counting the number of triangulations, $\mathcal{N}(N, l_i)$, with the topology of a sphere with n boundaries, and performing the sum and eventually taking the limit $a \to 0$, one can then explicitly find the partition function of two-dimensional quantum gravity.

我们将 Λ_0 和 $Z_i^{(0)}$ 称为裸的、未重整化的耦合常数，原因将在下文说明。通过对带有 n 个边界的球面拓扑的三角剖分数目 $\mathcal{N}(N,l_i)$ 计数，执行求和后最终取极限 $a \to 0$ ，就可以显式得到二维量子引力的配分函数。

Let us discuss the Wilsonian aspect of the above procedure. From the Wilsonian point of view, the continuum limit should not depend in a crucial way on precisely which class of triangulations one chooses. Similarly, one should be able not only to use triangles, but also squares, pentagons, etc. as building blocks, all with link lengths a . One then loses the unique piecewise geometry associated with a given graph T , but in the Wilsonian spirit one would still assume that for very large graphs one can make an identification like in Eq. (5):

下面我们来讨论上述过程的威尔逊观点。从威尔逊的角度来看，连续极限不应严重依赖于具体选择哪一类三角剖分。同理，我们不仅可以用三角形，还可以用正方形、五边形等作为构造单元，所有这些单元的链接长度均为 a 。此时我们会失去给定图 T 对应的唯一分片几何，但按照威尔逊精神我们仍假设，对于极大的图，可以得到类似式 (5) 的结果:

$$V(T) \propto N(T)\,a^2,\ L_i(T) \propto l_i(T)\,a, \tag{8}$$

where $N(T)$ denotes the number of polygons in the graph T and l_i the number of links of the i th boundary. This turns out to be true. For a particular set of graphs, so-called bipartite graphs (In this context we define the bipartite graphs as surfaces constructed by gluing together polygons with an even number of links and where also the boundary loops consist of an even number of links.) (again with the topology of a sphere with n boundaries), one can even find the corresponding generating function explicitly [15, 17] ,

其中 $N(T)$ 是图 T 中多边形的数量，l_i 是第 i 个边界的链接数。事实证明确实如此。对于一类特殊的图，即二分图 (在此我们将二分图定义为: 将偶数条边的多边形粘合得到的曲面，且边界环也由偶数个链接构成)(同样是带有 n 个边界的球面拓扑)，我们甚至可以显式得到对应的生成函数 [15, 17] ,

$$Z(g,z_i) = \left(\frac{1}{M_1(c^2,g)}\frac{d}{dc^2}\right)^{n-3}\frac{1}{2c^2M_1(g,c^2)}\prod_{i=1}^{n}\frac{c^2}{\left(z_i^2-c^2\right)^{3/2}},\ n \geq 3.$$

(9)

In this expression we have assigned the indeterminate $g_k = gw_k$ to each $2k$ -edged polygon which enters in the graph and an indeterminate $1/z_i$ to each link in the i th boundary. The relative weights of the polygons are $w_k \geq 0$ and

在该表达式中，我们将未定元 $g_k = gw_k$ 分配给图中每个 $2k$ 边多边形，将未定元 $1/z_i$ 分配给第 i 个边界的每个链接。多边形的相对权重为 $w_k \geq 0$ ，且

$$M_1(c^2,g) = \oint_C \frac{dz}{2\pi i}\frac{zV'(z)}{\left(z^2-c^2\right)^{3/2}},\ V'(z) = z - \sum_k g_k z^{2k-1}, \tag{10}$$

where the contour C encloses the cut $[-c, c]$ on the real axis and where we assume that only a finite, but in principle arbitrarily large, number of the w_k can be different from zero. Finally, the cut $[-c, c]$ is determined as a function of g by the following equation for $c^2(g)$

其中围道 C 包围实轴上的割线 $[-c, c]$，我们假设 w_k 中只有有限个 (但原则上可以任意大) 不等于零。最终，割线 $[-c, c]$ 作为 g 的函数，由 $c^2(g)$ 满足的下述方程确定

$$\oint_C \frac{dz}{2\pi i} \frac{zV'(z)}{\left(z^2 - c^2(g)\right)^{1/2}} = 2. \tag{11}$$

We present these explicit formulas because they tell us how to take the continuum limit of the lattice theory. We are interested in a limit where the number N of polygons goes to infinity. To each polygon we associate an indeterminate g , and $Z(g, z_i)$ has a convergent power expansion in g for small g . Large N will dominate when one reaches the radius of convergence of $Z(g, z_i)$. This occurs either when $M_1\left(c^2(g), g\right) = 0$ or when $c^2(g)$ ceases to be an analytic function of g . This happens to be at the same point g_0 . This $g_0\left(w_j\right)$ will be a function of the relative weights w_k , which in this discussion of convergence we consider fixed. Similarly, we might be interested in the situation where the lattice lengths l_i go to infinity. This happens by the same reasoning when $z_i = c(g)$, where (9) is nonanalytic in z_i . Denoting $z_0 = c(g_0), g_0\left(w_j\right)$ and $z_0\left(w_j\right)$ are critical points of our statistical system of graphs. By approaching these critical points according to

我们给出这些显式公式, 是因为它们说明了如何取格点理论的连续极限。我们关注多边形数量 N 趋于无穷的极限情况。我们为每个多边形分配一个不定元 g ,且 $Z(g, z_i)$ 在 g 很小时可以展开为关于 g 的收敛幂级数。当到达 $Z(g, z_i)$ 的收敛半径时, 大 N 项将占主导地位。这种情况发生在 $M_1\left(c^2(g), g\right) = 0$ 或者 $c^2(g)$ 不再是 g 的解析函数时, 而这两种情形恰好发生在同一点 g_0。该 $g_0\left(w_j\right)$ 是相对权重 w_k 的函数, 在本次收敛性讨论中我们将相对权重视为固定量。同理, 我们也可能关注格点长度 l_i 趋于无穷的情况, 根据相同的推导, 这会发生在 $z_i = c(g)$ 处, 此时式 (9) 关于 z_i 非解析。记 $z_0 = c(g_0), g_0\left(w_j\right)$ 和 $z_0\left(w_j\right)$ 是我们这个图统计系统的临界点。按照如下方式趋近这些临界点

$$g = g_0\left(w_j\right) \mathrm{e}^{-\Lambda a^2} = \mathrm{e}^{-\mu}, \ \frac{1}{z} = \frac{1}{z_0\left(w_j\right)} \mathrm{e}^{-Z_i a} = \mathrm{e}^{-\lambda_i}, \tag{12}$$

we can take the continuum limit of (9) by scaling $a \to 0$ and make contact with (4):

我们可以通过缩放 $a \to 0$ 对式 (9) 取连续极限，并得到式 (4):

$$Z(g, z_i) \propto a^{5-\frac{7n}{2}} \left(-\frac{d}{d\Lambda}\right)^{n-3} \left[\frac{1}{\sqrt{\Lambda}} \prod_{k=1}^{n} \frac{1}{\left(Z_i + \sqrt{\Lambda}\right)^{3/2}}\right] \propto a^{5-\frac{7n}{2}} Z(\Lambda, Z_i) \tag{13}$$

valid for $n \geq 3$. In particular we find from (13) and (4) by inverse Laplace transformation

该式对 $n \geq 3$ 成立。特别地，我们通过逆拉普拉斯变换从式 (13) 和式 (4) 得到

$$Z(V, L_i) \equiv \mathcal{N}(V, L_i) \propto V^{n-\frac{7}{2}} \sqrt{L_1 \cdots L_n} \mathrm{e}^{-(L_1 + \cdots + L_n)^2/4V}. \tag{14}$$

This formula is also valid for $n = 0, 1, 2$. From (12),(6), and (7), the relation between $\mathcal{N}(V, L_i)$ and $\mathcal{N}(N, l_i)$ is

该公式同样对 $n=0,1,2$ 成立。结合式 (12)、(6) 和 (7)，$\mathcal{N}(V,L_i)$ 和 $\mathcal{N}(N,l_i)$ 的关系为

$$\mathcal{N}(N,l_i) \propto \mathrm{e}^{\mu_0 N+\lambda_i^{(0)} l_i}\mathcal{N}(V,L_i),\ \mathrm{e}^{\mu_0}=g_0,\ \mathrm{e}^{\lambda_i^{(0)}}=\frac{1}{z_0}, \tag{15}$$

which shows that the number of generalized triangulations (bipartite graphs) with spherical topology and n boundaries grows exponentially with N , the number of polygons in the graphs. The number of graphs also grows exponentially with l_i , the number of boundary links. Equation (12) can be seen as additive renormalizations of the cosmological and boundary cosmological constants Λ_0 and $Z_i^{(0)}$:

这表明，具有球拓扑和 n 个边界的广义三角剖分 (二分图) 的数量随图中多边形的数量 N 指数增长，同时也随边界链接的数量 l_i 指数增长。式 (12) 可以看作是对宇宙学常数和边界宇宙学常数 Λ_0 和 $Z_i^{(0)}$ 的加法重整化:

$$\mu=\Lambda_0 a^2=\mu_0+\Lambda a^2,\ \lambda_i=Z_i^{(0)}a=\lambda_i^{(0)}+Z_i a. \tag{16}$$

The Wilsonian aspect of the above formulas is the following: we have an infinite-dimensional coupling constant space corresponding to $g_k \geq 0$. The critical surface is defined by $g_k = g_0(w_j)w_k$, where for given w_k , the $g_0(w_j)$ is the critical point discussed above. Thus, the critical surface has co-dimension 1 and approaching it for fixed w_k like in (12) leads to the same continuum theory. In this sense it is a beautiful example of Wilsonian universality, but one can ask: where is the divergent correlation length in the lattice theory, leading to this universality? This is especially interesting since this is a theory of quantum gravity, and as discussed above, we are integrating over to geometries which define length. We will discuss this in the next subsection.

上述公式的威尔逊观点如下: 我们存在一个对应于 $g_k \geq 0$ 的无穷维耦合常数空间。临界曲面由 $g_k=g_0(w_j)w_k$ 定义，对于给定的 w_k ，$g_0(w_j)$ 就是上文讨论的临界点。因此，临界曲面的余维数为 1，对固定 w_k 如式 (12) 那样趋近该曲面，就会得到同一个连续统理论。从这个意义上说，它是威尔逊普适性的一个绝佳范例，但我们不禁要问: 格点理论中导致这种普适性的发散关联长度在哪里? 这一点尤其有趣，因为这是量子引力理论，且正如上文所述，我们对定义长度的几何做了积分。我们将在下一小节讨论这个问题。

Let us end this subsection with a remark about the critical surface. We have restricted w_k to be larger than or equal to zero and with only a finite number of the w_k different from zero. If one relaxes these constraints, in particular the constraint that the w_k have to be positive, one can obtain different critical behaviors [37, 56] (which can be given the interpretation of matter systems coupled to quantum gravity). One obtains then a picture where a fine-tuning of the bare coupling constants to reach the critical surface might lead to different regions corresponding to different continuum theories. We will not pursue this possibility any further in this chapter, but only mention that the corresponding continuum quantum field theories are the so-called quantum Liouville theories with different central charge. These Liouville theories arise when quantizing two-dimensional Euclidean gravity coupled to conformal matter. Integrating out the matter fields, while using the conformal gauge for the metric, leads to an effective quantum field theory for the conformal factor of the metric, the Liouville quantum field theory, which depends on the central charge of the conformal matter field integrated out [49, 52, 53, 58] . The relation between the central charge c_L of the Liouville theory (which is also a conformal theory, although a somewhat special one) and the central charge c of the matter field is

$c_L = 26 - c$. The pure two-dimensional quantum gravity theory we have mainly discussed above corresponds in this notation to a conformal theory with central charge $c = 0$ and thus a Liouville theory with $c_L = 26$. The last 20 years has seen major progress in understanding and formulating the mathematics behind Liouville quantum gravity, and Chap. Thap. 74, ”Lessons from the Mathematics of Two-Dimensional Euclidean Quantum Gravity” in this section of the handbook, will describe this in detail.

我们在本小节最后谈谈临界曲面。我们已经限定 w_k 大于等于零，且只有有限个 w_k 非零。如果放宽这些约束，尤其是 w_k 必须为正的约束，就可以得到不同的临界行为 [37, 56]，这类行为可以解释为耦合量子引力的物质系统。由此我们得到的图景是: 裸耦合常数精细微调以达到临界曲面时，会得到对应不同连续理论的不同区域。本章我们不再进一步讨论这种可能性，仅提及对应的连续量子场论就是具有不同中心荷的所谓量子刘维尔理论。刘维尔理论产生于对耦合共形物质的二维欧几里得引力量子化。对物质场积分掉，并度规取共形规范后，会得到度规共形因子的有效量子场论，即刘维尔量子场论，它依赖于被积分掉的共形物质场的中心荷 [49, 52, 53, 58] 。刘维尔理论本身也是共形理论，尽管它有些特殊，其中心荷 c_L 与物质场中心荷 c 的关系为 $c_L = 26 - c$ 。我们上文主要讨论的纯二维量子引力理论，在此记号下对应中心荷为 $c = 0$ 的共形理论，因此是中心荷为 $c_L = 26$ 的刘维尔理论。过去 20 年间，学界对刘维尔量子引力的数学理解与构造取得了重大进展，本手册本章的第 74 章“二维欧几里得量子引力的数学启示”会详细对此进行介绍。

A Divergent Correlation Length in 2d EDT

二维 EDT 中的发散关联长度

In an ordinary quantum field theory in flat spacetime, a correlator is defined by

在平坦时空的普通量子场论中，关联函数定义为

$$\langle \phi(x)\phi(y)\rangle = \frac{\int \mathcal{D}\phi \mathrm{e}^{-S[\phi]}\phi(x)\phi(y)}{\int \mathcal{D}\phi \mathrm{e}^{-S[\phi]}}. \tag{17}$$

By translational and rotational invariance (which we will assume) $\langle \phi(x)\phi(y)\rangle$ is only a function of $|x-y|$, where x and y are spacetime points. We can take advantage of this by averaging over all points x and y separated by a distance $|x-y| = R$ and define

根据平移不变性和旋转不变性 (我们这里假设二者成立), $\langle \phi(x)\phi(y)\rangle$ 仅为 $|x-y|$ 的函数，其中 x 和 y 是时空点。我们可以利用这一性质，对所有间距为 $|x-y| = R$ 的点对 x 和 y 取平均，得到定义:

$$\langle \phi\phi\rangle_R = \frac{\int \mathcal{D}\phi \mathrm{e}^{-S[\phi]}\int dx \int dy \delta(|x-y|-R)\phi(x)\phi(y)}{\int \mathcal{D}\phi \mathrm{e}^{-S[\phi]}}, \tag{18}$$

where formally this average contains a factor V , the volume of spacetime, due to translational invariance. We can embed this definition of a correlation function in a quantum gravity theory

由于平移不变性，该平均形式上包含一个因子 V ，即时空体积。我们可以将这个关联函数的定义引入量子引力理论中

$$\langle\phi\phi\rangle_R = \frac{\int \mathcal{D}[g]\,\mathcal{D}\phi e^{-S[g,\phi]} \int dx \int dy \sqrt{g(x)g(y)}\delta\left(D_g(x,y)-R\right)\phi(x)\phi(y)}{\int \mathcal{D}[g]\,\mathcal{D}\phi e^{-S[g,\phi]}},$$

(19)

where $S[g,\phi]$ denotes the combined action of gravity and the field theory and where $D_g(x,y)$ is the geodesic distance between spacetime points labelled x and y . The correlation function (19) is diffeomorphism-invariant, but nonlocal. Since the gravity action contains a cosmological constant Λ , the average volume of spacetime will be finite and of order $1/\Lambda$. Contrary to (18) there is no infinite formal factor in the definition (19). We call R the quantum geodesic distance. Note that it will influence $\langle\phi\phi\rangle_R$ in a potentially much more radical way than the R in (18); when R is large compared to some appropriate power of $1/\Lambda$, it will define the shape of the whole universe in which we measure the correlation (For such a large R , the universe will be quite "elongated," because by definition at least two points have to be separated a geodesic distance R .). Thus, in some ways R is more like a new coupling constant in the theory, in the sense that the average shape of the universe depends on it.

其中 $S[g,\phi]$ 表示引力与场论的总作用量，$D_g(x,y)$ 是标记为 x 和 y 的两个时空点之间的测地线距离。式 (19) 的关联函数是微分同胚不变的，但是非局域的。由于引力作用量包含宇宙学常数 Λ，时空的平均体积是有限的，量级为 $1/\Lambda$ 。与式 (18) 不同，式 (19) 的定义中不存在无穷大形式因子。我们将 R 称为量子测地线距离。需要注意，它对 $\langle\phi\phi\rangle_R$ 的影响可能比式 (18) 中的 R 要彻底得多；当 R 远大于 $1/\Lambda$ 的某个适当幂次时，它会决定我们测量关联的整个宇宙的形状 (对于这么大的 R，宇宙会变得相当“伸长”，因为根据定义，至少有两个点的测地线距离为 R)。因此，从某种意义上说，R 更像是该理论中的一个新耦合常数，宇宙的平均形状依赖于它。

In the case of pure gravity, one has no external field $\phi(x)$, but one could consider curvature-curvature correlators (Chap. 78, "Spectral Observables and Gauge Field Couplings in Causal Dynamical Triangulations" in this section of the handbook, describes how to introduce curvature in lattice gravity theories.), or simply replace $\phi(x)$ by $1(x)$, which takes the value 1 for all x . This last choice is of interest since the correlator has a clear geometric interpretation and it can be explicitly calculated in the two-dimensional lattice gravity theory. We define the (unnormalized) two-point function corresponding to (19) with $\phi(x) = 1(x)$ as

对于纯引力的情况，不存在外场 $\phi(x)$，但我们可以考虑曲率-曲率关联函数 (本手册本章的第 78 章“因果动态三角剖分中的谱可观测量与规范场耦合”介绍了如何在格点引力理论中引入曲率)，或者直接将 $\phi(x)$ 替换为对所有 x 都取值为 1 的 $1(x)$。后一种选择很有意义，因为该关联函数具有清晰的几何解释，并且可以在二维格点引力理论中显式计算。我们将对应于式 (19)、取 $\phi(x) = 1(x)$ 的 (未归一化) 两点函数定义为

$$G_\Lambda(R) = \int \mathcal{D}[g]\, e^{-\Lambda V_g} \int dx \int dy \sqrt{g(x)g(y)}\delta\left(D_g(x,y)-R\right). \tag{20}$$

This is a formal continuum definition and requires a UV cutoff to define the path integral in (20). Again we use EDT and in addition now have to define the geodesic distance between the spacetime points x and y in formula (20) in the context of our triangulations. Let us for simplicity consider a triangulation constructed from equilateral triangles. As already mentioned, this triangulation can be viewed as a piecewise linear surface where the geometry is uniquely defined by assuming the triangles are flat in the interior. From such a piecewise linear triangulation, where one knows the length of each link, one can calculate $D_g(x,y)$. However, an approximate definition, convenient from a calculational point of view, is to define the graph distance between two links (One could also have chosen to define the graph distance between two vertices as the shortest link distance between the two vertices. As usual, from a Wilsonian point of view one should be led to the same continuum limit if it exists.) as the shortest distance, passing through centers of neighboring triangles; see Fig. 1.

这是形式上的连续统定义，需要紫外截断才能定义式 (20) 中的路径积分。我们再次使用 EDT，此外现在需要在我们的三角剖分框架下定义式 (20) 中两个时空点 x 和 y 之间的测地线距离。为简单起见，我们考虑由等边三角形构造的三角剖分。如前所述，这种三角剖分可以看作分段线性曲面，假设三角形内部是平坦的，就能唯一确定几何。从这种已知每条链接长度的分段线性三角剖分中，我们可以计算出 $D_g(x, y)$ 。不过，从计算角度出发，有一个方便的近似定义: 将两个链接之间的图距离定义为穿过相邻三角形中心的最短距离 (也可以选择将两个顶点之间的图距离定义为两顶点之间的最短链接距离。和通常情况一样，从威尔逊的观点来看，如果连续统极限存在，最终会得到相同的连续统极限)，参见图 1。

For generic, very large triangulations and for links with correspondingly large separation, we expect such a distance to be proportional to the "real" geodetic distance. Denote this graph distance between link ℓ and link ℓ' in a triangulation T of the sphere by $D_T(\ell, \ell')$. The lattice equivalent of (20) can be written as

对于一般的超大三角剖分，以及分离度相应很大的链接，我们预期这种距离与“真实的”测地线距离成正比。将球面三角剖分 T 中链接 ℓ 与链接 ℓ' 之间的图距离记为 $D_T(\ell, \ell')$ ，式 (20) 的格点对应形式可以写为

$$G_\mu(r) = \sum_T \mathrm{e}^{-\mu N(T)} \sum_{\ell,\ell'} \delta_{D_T(\ell,\ell'),r}. \tag{21}$$

Quite remarkably, one can combinatorially calculate the sum of these triangulations [11]. Close to the critical point μ_0 defined in (15), one obtains

非常值得注意的是，我们可以通过组合方法计算这些三角剖分的和 [11]。在式 (15) 定义的临界点 μ_0 附近，可以得到

$$G_\mu(r) \propto (\mu - \mu_0)^{3/4} \frac{\cos \sqrt[4]{\mu - \mu_0} r}{\sin^3 \sqrt[4]{\mu - \mu_0} r} \tag{22}$$

$$G_\mu(r) \propto (\mu - \mu_0)^{3/4} \mathrm{e}^{-2\sqrt[4]{\mu-\mu_0} r} \text{ for } r \gg \sqrt[4]{\mu - \mu_0}, \tag{23}$$

i.e., an exponential falloff with a correlation length $\xi(\mu) = 1/\sqrt[4]{\mu - \mu_0}$. Using (16), we can directly read off the continuum limit of (22), provided the geodesic distance R scales anomalously:

即关联长度为 $\xi(\mu) = 1/\sqrt[4]{\mu - \mu_0}$ 的指数衰减。若测地线距离 R 满足反常标度，利用式 (16) 我们可以直接读出式 (22) 的连续极限:

$$a^{-3/2} G_\mu(r) \propto G_\Lambda(R) = \Lambda^{3/4} \frac{\cos \sqrt[4]{\Lambda} R}{\sin^3 \sqrt[4]{\Lambda} R}, \ R = a^{1/2} r. \tag{24}$$

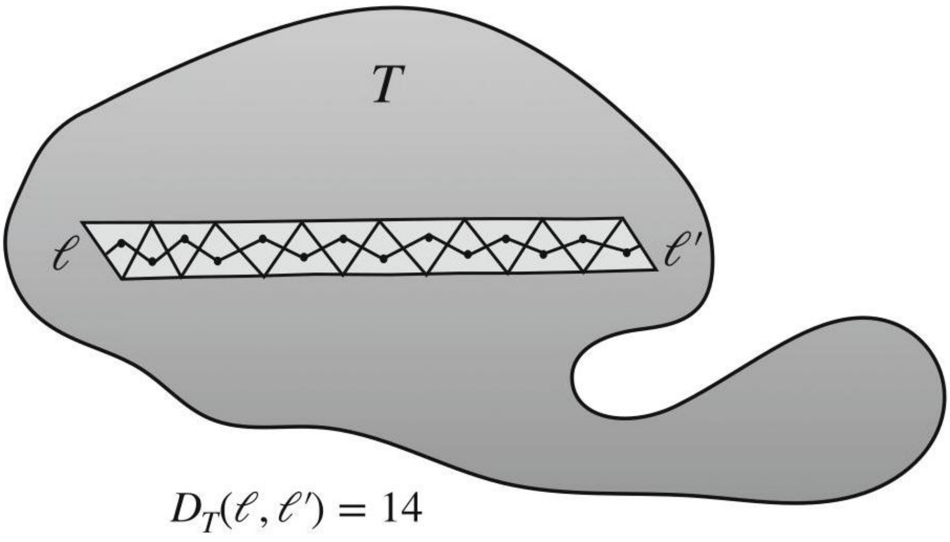

Fig. 1 A triangulation T with two links ℓ and ℓ' separated by a graph distance $D_T(\ell, \ell') = 14$

图 1 三角剖分 T ，其中两条链接 ℓ 和 ℓ' 的图距离为 $D_T(\ell, \ell') = 14$

Note that the anomalous dimension of R shows that the two-dimensional EDT quantum spacetime is fractal, with Hausdorff dimension $d_h = 4$ at all scales, as first realized in the seminal work [55]. Furthermore,

请注意，R 的反常维数表明，二维 EDT 量子时空是分形结构，正如开创性文献 [55] 首次指出的，它在所有尺度下的豪斯多夫维数均为 $d_h = 4$ 。此外，

$$\chi(\mu) = \sum_{r=1}^{\infty} G_\mu(r) = \text{ const. } - \frac{1}{6}\sqrt{\mu - \mu_0} + O(\mu - \mu_0) + \cdots \tag{25}$$

$$\equiv \text{ analytic } + \frac{1}{(\mu - \mu_0)^\gamma} + \cdots \tag{26}$$

where $\chi(\mu)$ denotes the susceptibility. The term $(\mu - \mu_0)^{-\gamma}$ is the leading nonanalytic term in the expansion of $\chi(\mu)$ around μ_0 , and γ is called the susceptibility exponent. These notations are inspired by the analogous notations used for spin-spin correlation functions in the theory of critical phenomena. From the definition (21) it follows that $\chi(\mu) \propto d^2 Z(\mu)/d\mu^2$, where $Z(\mu)$ is given by (6) with $n = 0$ (no boundaries). We thus obtain

其中 $\chi(\mu)$ 表示磁化率。$(\mu - \mu_0)^{-\gamma}$ 是 $\chi(\mu)$ 在 μ_0 附近展开的领头非解析项，γ 称为磁化率指数。这些记号借鉴了临界现象理论中描述自旋-自旋关联函数的类似记号。根据定义 (21) 可得 $\chi(\mu) \propto d^2 Z(\mu)/d\mu^2$ ，其中 $Z(\mu)$ 由式 (6) 在 $n = 0$ 条件下给出 (无边界)。因此我们得到

$$Z(\mu) \equiv \text{ analytic } + (\mu - \mu_0)^{2-\gamma} + \cdot, \ \gamma = -\frac{1}{2}, \tag{27}$$

a result which is consistent with (14) and (15). We have identified a divergent correlation length of two-dimensional EDT, and it is directly related to the fractal structure of the corresponding spacetime. The existence of this divergent correlation length explains why the Wilsonian picture works so well in this model.

A final remark concerns the quantum geodesic distance R which appears in the definition (20). Equation (24) shows how the choice of R will affect the general shape of the universe (as already mentioned above): for $R \gg 1/\Lambda^{1/4}$ it is a long tube of length R and cross section proportional to $\Lambda^{-3/4}$.

该结果与式 (14) 和 (15) 自洽。我们已经确定了二维 EDT 存在发散关联长度，且它直接对应于相应时空的分形结构。这种发散关联长度的存在解释了为何威尔逊图景在该模型中效果极佳。最后一点说明涉及定义 (20) 中出现的量子测地线距离 R 。式 (24) 表明，R 的选取会如何影响宇宙的整体形状 (正如上文已经提到的): 当满足 $R \gg 1/\Lambda^{1/4}$ 时，宇宙是一根长管，长度为 R ，横截面积正比于 $\Lambda^{-3/4}$ 。

The Generalized Two-Dimensional CDT Theory

广义二维 CDT 理论

As discussed above, the scaling limit for 2d EDT is essentially independent of the choice of weight w_n of the polygons, as long as the weights are nonnegative. In a Wilsonian context, a change of universality class is most likely related to a change of some global symmetry. The EDT formalism respects in a formal way the symmetry between space and (Euclidean) time, to a degree that it is unclear how one would actually rotate expressions like the two-point function $G_\Lambda(R)$ to spacetimes with a Lorentzian signature. Two-dimensional causal dynamical triangulation (CDT) is a regularization which takes the difference between space and time serious from the outset and insists on summing over spacetimes which have a well-defined time foliation. It is simplest to implement this in a discretized path integral if one assumes that space has the topology of a circle. Two neighboring spatial slices at discretized integer times k and $k+1$ then consist of l_k and l_{k+1} spatial links, and the two slices are connected by triangles with one spatial link and two time-like links, in such a way that the corresponding two-dimensional triangulation with the spatial slices at k and $k+1$ has the topology of a cylinder, as illustrated in Fig. 2. Clearly, one can in this way iteratively construct a two-dimensional triangulation with spatial slices at $k, k = 1, \ldots, s$, consisting of l_k links. This yields a cylinder with an ”entrance” spatial loop consisting of l_1 and an ”exit” spatial loop consisting of l_s links, as also shown in Fig. 2. Like in the EDT case, only the cosmological term will be important if we sum over piecewise linear manifolds with a fixed topology in the path integral. We can write, for a Lorentzian triangulation T_{lor} of the kind discussed,

如上所述, 二维欧氏动态三角化 (EDT) 的标度极限本质上与多边形权重 w_n 的选择无关, 只要权重非负即可。在威尔逊重整化的框架下，普适类的改变极有可能与某种整体对称性的改变相关。EDT 形式体系在形式上满足空间与 (欧氏) 时间的对称性, 以至于我们不清楚该如何将两点关联函数 $G_\Lambda(R)$ 这类表达式实际旋转到具有洛伦兹号差的时空上。二维因果动态三角化 (CDT) 是一种正则化方法，它从一开始就认真对待空间与时间的差异，坚持只对具有良定时间叶状结构的时空求和。如果假设空间具有圆周拓扑，那么最容易在离散路径积分中实现这一点。离散整数时间 k 和 $k+1$ 处的两个相邻空间切片分别包含 l_k 和 l_{k+1} 条空间边，两个切片通过包含一条空间边和两条类时边的三角形连接，由此得到的、包含 k 和 $k+1$ 处空间切片的二维三角化具有圆柱拓扑，如图 2 所示。显然，我们可以通过这种方式迭代构造出 $k, k = 1, \ldots, s$ 处包含空间切片的二维三角化，该三角化共包含 l_k 条边。最终得到一个圆柱，其“入口”空间环包含 l_1 条边，“出口”空间环包含 l_s 条边，如图 2 所示。与 EDT 的情况类似，在路径积分中对固定拓扑的分段线性流形求和时，只有宇宙学项会产生重要贡献。对于我们上述讨论的这类洛伦兹三角化 T_{lor} ，我们可以写出

$$S_{T_{\text{lor}}}(\Lambda,\alpha) = -\Lambda N(T_{\text{lor}})\frac{\sqrt{4\alpha+1}}{4}a^2,\ a_t^2 = -\alpha a^2,\ \alpha > 0. \tag{28}$$

In (28) we use the explicit area of a triangle with one spatial link of length $a_s^2 = a^2$ and two time-like links with $a_t^2 = -\alpha a^2$ (see [20, 31] for details). If $\alpha > 1/4$, we can perform an analytic continuation in the lower complex α -plane to negative $-\alpha$ such that

在式 (28) 中，我们使用了边长为 $a_s^2 = a^2$ 的单空间边与两条 $a_t^2 = -\alpha a^2$ 类时边构成的三角形的明确面积 (详见 [20, 31])。若 $\alpha > 1/4$ ，我们可以对复平面下半部分的 α 做解析延拓，得到负的 $-\alpha$ ，使得

$$S_{T_{\text{lor}}}(\Lambda,\alpha) \rightarrow S_{(T_{\text{lor}})}(\Lambda,-\alpha-i\varepsilon) = iS_{T_{\text{eucl}}}(\Lambda,\tilde{\alpha}), \tag{29}$$

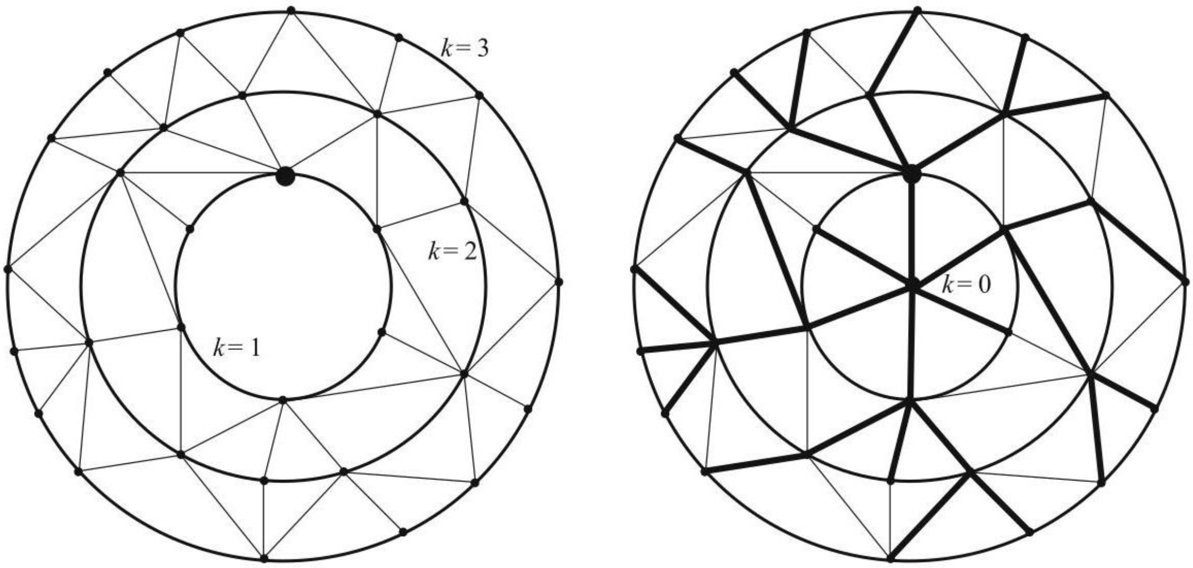

Fig. 2 Left figure: A CDT triangulation (represented as an annulus). Constant time slices corresponding to $k = 1, 2, 3$ are circles. A vertex (or the spatial link to the right of it) on the entrance loop $k = 1$ is marked. Right figure: The corresponding branched polymer (thick black links). An artificial vertex at $k = 0$ connected to each vertex at the $k = 1$ loop ensures a bijection between the CDT triangulations with boundaries at times $k = 1$ and $k = 3$ and so-called rooted branched polymers of height 3 (the root connects the vertex at $k = 0$ to the marked vertex at $k = 1$)

图 2 左图:CDT 三角化 (表示为环形)。对应 $k = 1, 2, 3$ 的等时切片是圆周。入口环 $k = 1$ 上标记了一个顶点 (以及它右侧的空间边)。右图: 对应的分支聚合物 (黑色粗边)。$k = 0$ 处的人工顶点连接 $k = 1$ 环上的每个顶点，保证了时间 $k = 1$ 和 $k = 3$ 处带边界的 CDT 三角化与高度为 3 的有根分支聚合物之间存在双射 (根将 $k = 0$ 处的顶点连接到 $k = 1$ 处的标记顶点)

where the Euclidian triangulation is denoted T_{eucl} and where

其中欧氏三角化记为 T_{eucl} ，且

$$S_{T_{\text{eucl}}}(\Lambda,\tilde{\alpha}) = \Lambda N(T_{\text{eucl}}) \frac{\sqrt{4\tilde{\alpha}-1}}{4} a^2 \quad \tilde{\alpha} = \alpha > \frac{1}{4}. \tag{30}$$

The inequality $\tilde{\alpha} > 1/4$ has the simple geometric interpretation that the sum of lengths of the two "time-like" triangle sides (i.e., $2\sqrt{\tilde{\alpha}}a$) has to be larger than the length a of the spacelike side of a triangle in the "flat" Euclidean triangles used in the rotated triangulation.

不等式 $\tilde{\alpha} > 1/4$ 有简单的几何解释: 在旋转变换后得到的“平坦”欧氏三角形中，两条“类时”三角形边长 (即 $2\sqrt{\tilde{\alpha}}a$) 之和必须大于类空间边 a 的长度。

Equation (29) is the "usual" formal relation between the Lorentzian and Euclidean actions, such that

式 (29) 是洛伦兹作用量与欧氏作用量之间“常规”的形式关系，满足

$$e^{iS_{T_{\text{lor}}}(\Lambda,\alpha)} = e^{-S_{T_{\text{eucl}}}(\Lambda,\tilde{\alpha})}, \quad \tilde{\alpha} = \alpha > \frac{1}{4}. \tag{31}$$

For each T_{lor} we perform the rotation to a corresponding T_{eucl} with the actions related by (29). The important point is that the class of triangulations $\{T_{\text{eucl}}\}$ obtained in this way is quite different from the class used in EDT. From now we will set $\alpha = 1$ since it only contributes a constant of proportionality to the action, where we have anyway already absorbed a factor of proportionality in Λ . We thus write, as in EDT(Again we assume, as in the EDT case, that a boundary link is marked on one of the boundary loops, to avoid symmetry factors occurring in the sum over triangulations.),

对每个 T_{lor} ，我们旋转到对应的 T_{eucl} ，二者的作用量满足关系式 (29)。重点在于，由此得到的三角剖分类 $\{T_{\text{eucl}}\}$ 与 EDT 中使用的三角剖分类截然不同。此后我们将设 $\alpha = 1$ ，因为它仅对作用量贡献一个比例常数，而我们 anyway 已经在 Λ 中吸收了一个比例因子。因此我们和 EDT 一样写作 (再次说明，我们和 EDT 情形一样假设其中一个边界环上标记了一条边界链，以避免三角剖分求和中出现对称因子):

$$S_T(\Lambda) = \Lambda N(T) a^2 = \mu N(T), \ \mu = \Lambda_0 a^2, \tag{32}$$

$$Z(\mu,\lambda_1,\lambda_s) = \sum_T e^{-\mu N(T) - \lambda_1 l_1 - \lambda_s l_s} \tag{33}$$

$$= \sum_{N,l_1,l_s} e^{-\mu N(T) - \lambda_1 l_1(T) - \lambda_s l_s(T)} \mathcal{N}(N,l_0,l_s), \tag{34}$$

where the summation is over the triangulations described above, which have the topology of a cylinder, with s spatial slices, where slice 1 consists of l_1 spatial links and slice s of l_s spatial links.

其中求和对上述三角剖分进行，这些三角剖分具有柱面拓扑，包含 s 个空间切片，切片 1 由 l_1 条空间链构成，切片 s 由 l_s 条空间链构成。

The two-dimensional CDT model (and related models) can be solved analytically [3, 7, 51] , and rather surprisingly the critical exponents of the model agree with corresponding critical exponents of tree graphs or so-called branched polymers. Later it was understood that this is not a coincidence [46], but that there exists a bijective map of the CDT surface graphs onto so-called rooted branched graphs of height $s+1$, as illustrated in Fig. 2. This insight highlights the importance of treelike structures in graphs relevant to quantum gravity. Chapter 75, ”From Trees to Gravity” in this section of the handbook is dedicated the study of such treelike graphs.

> 二维 CDT 模型 (及相关模型) 可解析求解 [3, 7, 51]，相当出人意料的是，该模型的临界指数与树图即所谓分支聚合物的对应临界指数一致。后来人们认识到这并非巧合 [46]，CDT 曲面图存在到高度为 $s+1$ 的有根分支图的双射，如图 2 所示。这一观点凸显了树状结构在量子引力相关图中的重要性。本手册本篇的第 75 章《从树到引力》专门研究这类树状图。

If we define a slightly modified CDT graph by connecting all vertices at time slice $k = 1$ to a single vertex at a new time slice at $k = 0$ and all vertices at time slice $k = s$ to a single vertex at a new time slice at $s + 1$, then the graph distance (which we here define to be the shortest link distance) between any vertices will be less or equal to $s + 1$. If we start at the vertex at time 0, then the only vertex where the graph distance to the starting vertex is a local maximum is the vertex at $s + 1$ (and the local maximum is in this case also a global maximum). From a graph point of view this is a rather special situation and one can generalize it to include graphs where a finite number of vertices have a local maximum distance to a starting vertex, even in the limit where the number of vertices goes to infinity. This is the setup of generalized CDT: starting from a vertex or a spatial entrance loop, one moves forward in ”proper time,” which is defined as the graph distance from the vertex or the entrance loop (and in the continuum by the geodesic distance from the entrance loop). On the way to the exit spatial loop (or loops), space can branch into several disconnected spatial universes. The ones that do not end in exit loops vanish into the ”vacuum.” The distances of these vacuum points to the entrance loop are then local maxima, and the spatial loops that in this way disappear into the vacuum are called baby universes. For graphs consisting of a finite number of vertices, there is no real difference between the graphs used in EDT and the ones used in generalized CDT, but the crucial difference comes from requiring that when the number of vertices goes to infinity, the number of baby universes stays finite. This will then also be true in the continuum limit and is in contrast to the EDT situation, where the fractal structure with Hausdorff dimension $d_h = 4$ implies that infinitely many baby universes (but most of them with infinitesimal volume) are created in the continuum limit. In the case of generalized CDT one finds $d_h = 2$. Again the discretized model can be solved analytically and one can take the continuum limit (Rather amazingly, it is possible to solve the model directly in the continuum simply by using that the number of baby universes is finite [23, 26, 27]) in much the same way as was done in the EDT case [1]. In particular, one can find the continuum version of the generalized CDT two-point function

如果我们定义一个稍作修改的 CDT 图: 将时间切片 $k = 1$ 的所有顶点连接到新时间切片 $k = 0$ 的单个顶点，再将时间切片 $k = s$ 的所有顶点连接到新时间切片 $s+1$ 的单个顶点，那么任意顶点之间的图距离 (我们在此将其定义为最短链距离) 将小于等于 $s+1$ 。如果我们从时间 0 处的顶点出发，那么图距离等于起点距离局部最大值的顶点只有 $s+1$ 处的顶点 (在该情形下这个局部最大值同时也是全局最大值)。从图论的角度看这是相当特殊的情况，可以将其推广: 即使在顶点数趋于无穷的极限下，也允许有限个顶点到起点的距离为局部最大值。这就是广义 CDT 的框架: 从一个顶点或一条空间入口环出发，沿着” 固有时” 向前移动，固有时定义为到该顶点或入口环的图距离 (连续情形下则是到入口环的测地线距离)。在通往出口空间环 (或多个出口环) 的途中，空间可以分支为多个不连通的空间宇宙，其中不终止于出口环的分支会消失到” 真空” 中。这些真空点到入口环的距离就是局部最大值，这种消失到真空的空间环被称为婴儿宇宙。对于顶点数有限的图，EDT 使用的图和广义 CDT 使用的图没有实质差异，关键差异来自于顶点数趋于无穷时的要求: 婴儿宇宙的数量保持有限。这一点在连续极限下也成立，这与 EDT 的情况不同，EDT 中豪斯多夫维数为 $d_h = 4$ 的分形结构意味着，连续极限下会产生无穷多个婴儿宇宙 (其中大多数体积无穷小)。在广义 CDT 中我们得到 $d_h = 2$ 。同样，离散化模型可以解析求解，也可以取连续极限 (相当惊人的是，可以直接在连续情形下求解该模型，只需利用婴儿宇宙数量有限这一性质 [23, 26, 27]) ， 方法和 EDT 情形中 [1] 的做法大致相同。特别地，我们可以得到广义 CDT 两点关联函数的连续形式

$$G_\Lambda(\tau) = \frac{\sum^3}{\Theta} \frac{\sum \sin \sum \tau + \Theta \cos \sum \tau}{\left(\sum \cos \sum \tau + \Theta \sin \sum \tau\right)^3} \tag{35}$$

where

其中

$$\sum = \sqrt{\Lambda} H\left(\frac{G_b}{\Lambda^{3/2}}\right),\ H(0) = 1;\ \Theta = \sqrt{\Lambda} F\left(\frac{G_b}{\Lambda^{3/2}}\right),\ F(0) = 1.$$

(36)

The functions $H(x)$ and $F(x)$ have a power expansion in x , with a radius of convergence $2/3^{3/2}$. A new coupling constant denoted G_b has appeared in (36). It is the coupling constant for a spatial universe to split into two spatial universes. When $G_b/\Lambda^{3/2} \to 0$, we get back to ordinary CDT. When $G_b/\Lambda^{3/2} \to 2/3^{3/2}$,(35) and (36) cease to be valid and one can show that the number of baby universes goes to infinity, indicating that one has a phase transition to ordinary EDT gravity. We have in (35) denoted the geodesic distance entering in the two-point function by τ (proper time) rather than the R used in (24), to emphasize the origin as a proper time in Lorentzian CDT. Equation (35) looks superficially like a generalization of Eq. (24). However, the important point is that we have $\sqrt{\Lambda}\tau$ as an argument, while in (24) $\sqrt[4]{\Lambda}R$ appears as an argument, capturing the difference in Hausdorff dimension for the two ensembles of geometries. We still have a perfect Wilsonian picture for generalized CDT as embedded in EDT [28,34]. In EDT one can introduce an additional dimensionless coupling g_b , which controls the creation of baby universes, such that for small values of g_b the creation of baby universes is suppressed. One can show that for any finite value of g_b the critical behavior of the system is still that of two-dimensional Euclidean gravity. However, if g_b is scaled to zero at the same time as one approaches the critical surface in the following way (which is a generalization of (16))

函数 $H(x)$ 和 $F(x)$ 可在 x 处进行幂展开，收敛半径为 $2/3^{3/2}$ 。式 (36) 中出现了一个新的耦合常数，记为 G_b ，它是空间宇宙分裂为两个空间宇宙的耦合常数。当 $G_b/\Lambda^{3/2} \to 0$ 时，我们回到普通 CDT 理论。当 $G_b/\Lambda^{3/2} \to 2/3^{3/2}$ 时，式 (35) 和 (36) 不再成立，并且可以证明婴儿宇宙的数量趋于无穷，这表明系统发生了向普通 EDT 引力的相变。为了强调它是洛伦兹 CDT 中固有时间的起源，我们在式 (35) 中将两点函数中的测地线距离记为 τ (固有时间)，而非式 (24) 中使用的 R 。式 (35) 表面上看是式 (24) 的推广，但重点在于我们用 $\sqrt{\Lambda}\tau$ 作为自变量，而式 (24) 中以 $\sqrt[4]{\Lambda}R$ 作为自变量，这体现了两个几何系综在豪斯多夫维数上的差异。对于嵌入 EDT 的广义 CDT，我们仍然有完整的威尔逊图景 [28,34]。在 EDT 中可以引入一个额外的无量纲耦合常数 g_b ，它控制婴儿宇宙的产生，当 g_b 取值较小时，婴儿宇宙的产生被抑制。可以证明，对于 g_b 的任意有限值，系统的临界行为仍然符合二维欧几里得引力的临界行为。但若 g_b 在趋近临界曲面的同时按如下方式缩放至零 (这是式 (16) 的推广)

$$g_b = G_b a^3,\ \mu = \mu_0 + \Lambda a^2,\ \lambda_i = \lambda_i^{(0)} + Z_i a, \tag{37}$$

one obtains generalized CDT with continuum coupling constants Λ, Z_i , and G_b . From a Wilsonian point of view we have an infinite-dimensional critical surface and on this critical surface a subspace where $g_b = 0$. On this critical subspace there is an asymmetry between ”space” and ”time,” the same asymmetry that was put in by hand in the original simple CDT model and that was not present in the EDT model. Approaching the subspace as in (37), one will obtain the continuum limit corresponding to generalized CDT, while approaching the critical surface at a point where $g_b > 0$, in the way described by (16), leads to the continuum limit of EDT.

我们就得到了带有连续耦合常数 Λ, Z_i 和 G_b 的广义 CDT。从威尔逊的观点来看，我们得到一个无穷维临界曲面，该曲面上存在一个满足 $g_b = 0$ 的子空间。在这个临界子空间上，“空间”和“时间”之间存在不对称性，这种不对称性就是最初简单 CDT 模型中人为引入、而 EDT 模型中不具备的那种不对称性。按照式 (37) 的方式趋近该子空间，就会得到对应广义 CDT 的连续极限；而按照式 (16) 描述的方式，在满足 $g_b > 0$ 的点趋近临界曲面，就会得到 EDT 的连续极限。

Hořava-Lifshitz gravity is a continuum theory where one demands that spacetime has a time foliation and that the theory is invariant under spatial diffeomor-phisms and time redefinitions [54]. Chap. 77, ”CDT and Hořava-Lifshitz QG in Two Dimensions” discusses (among other topics) the relation between two-dimensional CDT and two-dimensional Hořava-Lifshitz gravity (see also [32]). In the case of higher-dimensional CDT no such relation is known to exist.

霍拉瓦-里夫希茨引力是一种要求时空满足时间叶化，且理论在空间微分同胚和时间重定义下保持不变的连续统理论 [54]。第 77 章“二维 CDT 与二维霍拉瓦-里夫希茨量子引力”除其他主题外，还讨论了二维 CDT 与二维霍拉瓦-里夫希茨引力之间的关系 (另见 [32])。目前已知高维 CDT 中不存在这种关系。

Four-Dimensional EDT

四维 EDT

While two-dimensional EDT and CDT have a clear Wilsonian interpretation where continuum limits can be defined and continuum correlators can be calculated analytically, the situation is more complicated when one wants to generalize the EDT and CDT formalism to higher-dimensional gravity. First, higher-dimensional gravity is non-renormalizable, and the curvature term that dropped out in two-dimensional quantum gravity is now expected to play a key role. In the Wilsonian context of asymptotic safety one needs a nontrivial UV fixed point of the lattice theory if conventional renormalization group logic applies and if one wants the lattice theory to define a quantum continuum theory at all scales. Second, contrary to the situation in two dimensions, there is presently no way we can solve the lattice theory analytically. We have to rely on Monte Carlo simulations of the path integral. This implies that we have to use an action with Euclidean signature, since the Monte Carlo simulations need the exponential of the action to have a probability interpretation. Unfortunately, the Euclidean four-dimensional continuum Einstein-Hilbert action is unbounded from below and this will be true also for the lattice action when the lattice volume becomes infinite. Only the measure term in the path integral may save us, if we want to restrict ourselves to the Einstein-Hilbert term as the classical action appearing in the path integral. Alternatively, one could include higher curvature terms in the classical action. Finally, we have to be able to find second- or higher-order phase transitions for the lattice theory, as discussed above.

> 尽管二维 EDT 与 CDT 具有清晰的威尔逊诠释，可定义连续极限且能解析计算连续关联函数，但若要将 EDT 与 CDT 形式推广至高维引力，情况会复杂得多。首先，高维引力不可重整，二维量子引力中消失的曲率项，如今被认为会发挥关键作用。在渐近安全的威尔逊框架下，若常规重整化群逻辑成立，且我们希望格点理论能在所有尺度上定义量子连续理论，就要求格点理论存在一个非平凡紫外不动点。其次，与二维情况不同，目前我们无法解析求解该格点理论，必须依赖路径积分的蒙特卡罗模拟。这意味着我们需要使用欧几里得符号的作用量，因为蒙特卡罗模拟要求作用量的指数具有概率诠释。遗憾的是，欧几里得四维连续爱因斯坦-希尔伯特作用量下有界，当格点体积趋于无穷时，格点作用量也同样下无界。如果我们仅将爱因斯坦-希尔伯特项作为路径积分中的经典作用量，那么只有路径积分中的测度项能挽救这一问题；当然也可以选择在经典作用量中加入更高阶曲率项。最后，正如前文讨论，我们还必须能为该格点理论找到二级或更高级相变。

Let us define the lattice theory. We follow the two-dimensional theory and consider four-dimensional piecewise linear geometries, constructed by gluing together building blocks consisting of four simplices, where all link lengths are equal to a , which then serves as our UV cutoff. The only restriction on the gluing is that the gluing locally is such that one has a (piecewise linear) manifold and that the topology of this piecewise linear manifold is fixed (One can study more general models where one relaxes the constraint that the gluing should result in a piecewise linear geometry or that the topology of the manifolds should be fixed. It is possible to formulate such a generalized gluing procedure in different ways, and starting with the articles [16, 66], these models are denoted tensor models. If we discuss the gluing of d -dimensional simplices, the number of tensor indices are equal to the number of $d-1$ -dimensional subsimplices in the d -dimensional simplices which constitute the building blocks. For $d = 2$ we have tensors of rank 2, i.e., matrices, and two-dimensional gravity has indeed been studied using matrix models, starting with the work of David [48].). Most studied is the simplest topology, the four-sphere S^4 , and we will limit ourselves to discussing this case. We consider the four-simplices as flat in the interior. Viewed as a piecewise linear continuum manifold, all geodesic distances are well-defined, and thus, the geometry of such a triangulation is also fixed without specifying a coordinate system. Summing over combinatorially inequivalent triangulations then results in a summation over a certain class of piecewise linear geometries, and the hope is that in the limit where the lattice spacing $a \to 0$, this summation will in some sense be a good representation of the integration over continuous geometries (of which the piecewise linear geometries constitute a subset that is hopefully dense with respect to a [still] unknown measure).

下面我们来定义这个格点理论。我们沿用二维理论的框架，研究由四维单形作为构件粘合而成的四维分段线性几何，所有链接长度均等于 a ，该长度即为我们的紫外截断。对粘合的唯一约束是: 局部粘合后得到一个 (分段线性) 流形，且该分段线性流形的拓扑固定 (人们也研究更一般的模型，这类模型放宽了粘合必须得到分段线性几何、流形拓扑必须固定的约束。这类广义粘合过程可以用多种不同方式表述，自文献 [16, 66] 起，这类模型被称为张量模型。当我们讨论 d 维单形的粘合时，张量指标的数量等于构成构件的 d 维单形中 $d-1$ 维子单形的数量。对于 $d=2$ ，我们得到秩为 2 的张量，即矩阵，而自 David 的工作 [48] 开始，人们确实使用矩阵模型研究了二维引力。) 研究最多的是最简单的拓扑，即四维球面 S^4 ，本文我们也仅讨论这一情况。我们假设四维单形内部是平直的。作为分段线性连续流形，其所有测地线距离都是良定义的，因此无需指定坐标系即可确定这类三角剖分的几何。对不同组合不等价的三角剖分求和，等价于对某一类分段线性几何求和，我们期望在格点间距 $a \to 0$ 的极限下，该求和能在某种意义上很好地表示对连续几何的积分 (分段线性几何是连续几何的子集，我们希望它相对于 [尚未可知的] 测度是稠密的)。

An obvious question is how to represent the curvature term present in the Einstein-Hilbert action. Regge showed how to define curvature locally on a d -dimensional piecewise linear manifold [65] constructed from d -simplices σ_j^d , by locating it on $(d-2)$ -dimensional subsimplices σ_i^{d-2} . The d -simplices σ^d sharing a $(d-2)$ -dimensional subsimplex σ^{d-2} have dihedral angles (For a given d -simplex, any of its $(d-2)$ -subsimplices will be the intersection of precisely two of its $(d-1)$ -subsimplices, and the angle between these two $(d-1)$ -subsimplices is called the dihedral angle (which is an angle for any $d \geq 2$).) $\theta\left(\sigma^{d-2}, \sigma^d\right)$, related to this subsimplex. If the spacetime was flat, these dihedral angles would add to 2π . The so-called deficit angle $\varepsilon_{\sigma^{d-2}}$, associated with the subsimplex σ^{d-2} , is defined by

一个很自然的问题是: 如何表示爱因斯坦-希尔伯特作用量中的曲率项。里奇证明了，可以通过将曲率定位在 $(d-2)$ 维子单形 σ_i^{d-2} 上，从而在由 d 维单形 σ_j^d 构造出的 d 维分段线性流形 [65] 上局域定义曲率 $\theta\left(\sigma^{d-2}, \sigma^d\right)$。共享同一个 $(d-2)$ 维子单形 σ^{d-2} 的 d 维单形 σ^d 存在与该子单形相关的二面角 (对任意给定的 d 维单形，它的任意一个 $(d-2)$ 维子单形都是其两个 $(d-1)$ 维子单形的交，这两个 $(d-1)$ 维子单形之间的夹角就称为二面角，对任意 $d \geq 2$ 而言它都是一个角度。)。若时空是平坦的，这些二面角之和等于 2π 。与子单形 σ^{d-2} 关联的所谓亏缺角 $\varepsilon_{\sigma^{d-2}}$ 定义为

$$\varepsilon_{\sigma^{d-2}} = 2\pi - \sum_{\{\sigma^d \mid \sigma^{d-2} \in \sigma^d\}} \theta\left(\sigma^{d-2}, \sigma^d\right). \tag{38}$$

The deficit angle is the angle by which a vector will be rotated when parallel-transported locally around the $(d-2)$ -simplex in the piecewise linear geometry in the subspace perpendicular to the $d-2$ -dimensional simplex. Regge showed that the curvature action and the volume term associated to the piecewise linear manifold M can be written as

亏缺角是指，在分段线性几何中，一个矢量绕垂直于 $d-2$ 维单形的子空间内的 $(d-2)$ 维单形做局域平行运输后，转过的角度。里奇证明了，分段线性流形 M 对应的曲率作用量与体积项可以写为

$$\int_M \mathrm{d}^d \xi \sqrt{|g(\xi)|} R(\xi) = 2 \sum_{\sigma^{d-2}} \varepsilon_{\sigma^{d-2}} V_{\sigma^{d-2}}, \quad \int_M \mathrm{d}^d \xi \sqrt{|g(\xi)|} = \sum_{\sigma^d} V_{\sigma^d},$$

(39)

where $V_{\sigma^{d-2}}$ denotes the volume of the subsimplex σ^{d-2} and V_{σ^d} the volume of the simplex σ^d (in the case where $d=2$, we define $V_{\sigma^{d-2}}=1$).

其中 $V_{\sigma^{d-2}}$ 是子单形 σ^{d-2} 的体积，V_{σ^d} 是单形 σ^d 的体积 (当 $d = 2$ 时，我们定义 $V_{\sigma^{d-2}} = 1$)。

For the piecewise linear geometries used in EDT, this expression simplifies enormously since all dihedral angles are identical, all $(d-2)$ -volumes are the same, and all d -volumes are also equal. In the four-dimensional case, which has our main interest, we have

对于动态三角剖分 (EDT) 中使用的分段线性几何，这个表达式会大幅简化，因为所有二面角都相同，所有 $(d-2)$ 体积都相等，所有 d 体积也相等。对于我们主要关注的四维情形，我们有

$$\theta\left(\sigma^2, \sigma^4\right) = \arccos\frac{1}{4},\ V_{\sigma^2} = \frac{\sqrt{3}}{2}a^2,\ V_{\sigma^4} = \frac{\sqrt{5}}{96}a^4. \tag{40}$$

The Einstein-Hilbert action for a given triangulation T of a closed four-dimensional manifold in EDT can then be written as

因此，EDT 中闭四维流形给定三角剖分 T 对应的爱因斯坦-希尔伯特作用量可以写为

$$S_M(G, \Lambda) = \frac{1}{16\pi G}\int d^4\xi\sqrt{g(\xi)}\left(-R(\xi) + 2\Lambda\right) \rightarrow$$

$$S_T(\kappa_2, \kappa_4) = -\kappa_2 N_2(T) + \kappa_4 N_4(T), \tag{41}$$

$$\kappa_2 = \frac{1}{8G}\frac{\sqrt{3}a^2}{2},\ \kappa_4 = \frac{2\Lambda}{16\pi G}\frac{\sqrt{5}a^4}{96} + 20\arccos\left(\frac{1}{4}\right)\frac{1}{16\pi G}\frac{\sqrt{3}a^2}{2}, \tag{42}$$

where $N_2(T)$ denotes the number of two-simplices and $N_4(T)$ the number of four-simplices in the triangulation T . From the so-called Dehn-Sommerville relations for a closed four-dimensional triangulation T , one has that $N_2(T) = 2N_0(T) + 2N_4(T) - 2\chi$, where $N_0(T)$ denotes the number of vertices in the triangulation and χ is the Euler characteristic of the triangulation. The Euler characteristic only depends on the topology of the triangulation. Thus, (41) can be written as

其中 $N_2(T)$ 是三角剖分 T 中二维单形的数量，$N_4(T)$ 是四维单形的数量。根据闭四维三角剖分 T 的德恩-索默维尔关系，可得 $N_2(T) = 2N_0(T) + 2N_4(T) - 2\chi$ ，式中 $N_0(T)$ 是三角剖分中顶点的数量，χ 是三角剖分的欧拉示性数。欧拉示性数仅依赖于三角剖分的拓扑，因此式 (41) 可以写为

$$S_T(k_0, k_4) = -k_0 N_0(T) + k_4 N_4(T) + k_0\chi,\ k_0 = 2\kappa_2,\ k_4 = \kappa_4 - 2\kappa_2 \tag{43}$$

where the χ -term is usually ignored since we consider triangulations with a fixed topology.

式中 χ 项通常可以忽略，因为我们考虑固定拓扑的三角剖分。

The EDT partition function of four-dimensional quantum gravity is now obtained by summing over triangulations with the action given by (41) (or (43)):

四维量子引力的 EDT 配分函数可以通过对作用量由式 (41)(或式 (43)) 给出的所有三角剖分求和得到:

$$Z(\kappa_2,\kappa_4)=\sum_T \frac{1}{C_T}\mathrm{e}^{\kappa_2 N_2(T)-\kappa_4 N_4(T)}=\sum_{N_4,N_2}\mathrm{e}^{\kappa_2 N_2-\kappa_4 N_4}\mathcal{N}(N_2,N_4), \tag{44}$$

where the summation is over all abstract triangulations (We use here the notation ”abstract triangulation” to emphasize that although we have viewed the triangulations as piecewise linear manifolds and have introduced a link length a as a UV cutoff, in the summation (44) only the labelling as (abstract) triangulations is important. The other aspects will be important when we discuss a continuum limit.) T of a given four-dimensional manifold, where C_T is a symmetry factor (the order of the automorphism group of the triangulation T) and where $\mathcal{N}(N_2,N_4)$ denotes the number of such triangulations with a fixed number of two-simplices and four-simplices, N_2 and N_4 , respectively. As was the case in two dimensions, the partition function is entirely combinatorial: $Z(\kappa_2,\kappa_4)$ is the generating function (with indeterminates e^{κ_2} and $e^{-\kappa_4}$) for the number of four-dimensional triangulations with a given topology (here S^4) and a given number of four-simplices and two-simplices. It is truly remarkable that four-dimensional quantum gravity in this way is purely ”entropic.” Unfortunately, it is not yet possible to perform this counting analytically. This leaves us presently with Monte Carlo simulations if we want to study the partition function (44).

其中求和遍历所有抽象三角剖分 (我们在此使用“抽象三角剖分”这一名称是为了强调: 尽管我们将三角剖分视为分段线性流形，并引入了链接长度 a 作为紫外截断，但在求和式 (44) 中，只有作为 (抽象) 三角剖分的标记是重要的，其他方面会在我们讨论连续极限时才发挥作用。) T 针对给定四维流形，其中 C_T 是对称因子 (即该三角剖分自同构群的阶 T)，$\mathcal{N}(N_2,N_4)$ 表示固定二维单形和四维单形数量的这类三角剖分的数量，二者分别对应 N_2 和 N_4 。与二维的情况一样，配分函数完全是组合性的: $Z(\kappa_2,\kappa_4)$ 是对应给定拓扑 (此处为 S^4)、给定四维单形和二维单形数量的四维三角剖分数量的生成函数 (不定元为 e^{κ_2} 和 $e^{-\kappa_4}$)。值得注意的是，以此方式构建的四维量子引力纯粹是“熵性的”。遗憾的是，目前还无法通过解析方法完成这种计数。如果我们想要研究配分函数 (44)，目前只能采用蒙特卡洛模拟。

In view of the unboundedness of the Euclidean Einstein-Hilbert action, the first obvious question one can ask is whether $Z(\kappa_2,\kappa_4)$ is at all well-defined for any values of κ_2 and κ_4 . Let us perform the summation over N_2 in (44),

鉴于欧几里得爱因斯坦-希尔伯特作用量无界，我们首先会想到一个显而易见的问题: $Z(\kappa_2,\kappa_4)$ 对任意 κ_2 和 κ_4 的取值而言是否是良定义的。我们来对 (44) 中的 N_2 执行求和,

$$Z(\kappa_2,\kappa_4)=\sum_{N_4}\mathrm{e}^{-\kappa_4 N_4}\mathcal{N}_{\kappa_2}(N_4),\ \mathcal{N}_{\kappa_2}(N_4)=\sum_{N_2}\mathrm{e}^{\kappa_2 N_2}\mathcal{N}(N_2,N_4). \tag{45}$$

If $\mathcal{N}_{\kappa_2}(N_4)$ is exponentially bounded as a function of N_4 , i.e., if there exists a constant $\kappa_4^c(\kappa_2)$ such that

如果 $\mathcal{N}_{\kappa_2}(N_4)$ 作为 N_4 的函数是指数有界的，即存在常数 $\kappa_4^c(\kappa_2)$ 满足

$$\mathcal{N}_{\kappa_2}(N_4)\leq \mathrm{e}^{(\kappa_4^c(\kappa_2)+\varepsilon)N_4},\ \ \text{for all }\varepsilon>0,\ N_4>N_4(\varepsilon), \tag{46}$$

then there is a line $(\kappa_2,\kappa_4^c(k_2))$ in the κ_2,κ_4 coupling-constant plane, such that $Z(\kappa_2,\kappa_4)$ is well-defined and convergent for $\kappa_4>\kappa_4^c(\kappa_2)$. It is easy to prove that if (46) is valid for $\kappa_2=0$, then it is valid for all κ_2 (with a $\kappa_4^c(\kappa_2)$ depending on κ_2). However, there is no proof that $\mathcal{N}_0(N_4)$ is exponentially bounded. Computer simulations indicate that it is the case [5, 43] and in the following we will assume so. The physics of (45) is all hidden in $\mathcal{N}_{\kappa_2}(N_4)$, which according to our assumptions can be written as

那么在 κ_2, κ_4 耦合常数平面中存在一条线 $(\kappa_2, \kappa_4^c(k_2))$ ，使得 $Z(\kappa_2, \kappa_4)$ 对 $\kappa_4 > \kappa_4^c(\kappa_2)$ 而言是良定义且收敛的。不难证明，如果 (46) 对 $\kappa_2 = 0$ 成立，那么它对所有 κ_2 都成立 (其中 $\kappa_4^c(\kappa_2)$ 依赖于 κ_2)。但目前还没有证明能说明 $\mathcal{N}_0(N_4)$ 是指数有界的。计算机模拟表明该结论成立 [5, 43]，在下文中我们将以此为前提。(45) 的物理性质全部隐藏在 $\mathcal{N}_{\kappa_2}(N_4)$ 中，根据我们的假设它可以写为

$$\mathcal{N}_{\kappa_2}(N_4) = e^{\kappa_4^c(k_2)N_4} H_{\kappa_2}(N_4), \ H_{\kappa_2}(N_4) \text{ subleading in } N_4, \tag{47}$$

which implies that the nontrivial continuum physics is to be found in the subleading function $H_{\kappa_2}(N_4)$. One then obtains

这意味着非平凡的连续物理蕴含在次领头阶函数 $H_{\kappa_2}(N_4)$ 中。由此我们得到

$$Z(\kappa_2, \kappa_4) = \sum_{N_4} e^{-(\kappa_4 - \kappa_4^c(k_2))N_4} H_{\kappa_2}(N_4). \tag{48}$$

Given two triangulations $T(N_4)$ and $T(N_4')$, there exist local changes in the triangulation T_4 (the so-called Pachner moves) that, when applied a finite number of times, will bring us from $T(N_4)$ to $T(N_4')$. These Pachner moves [62] are used in the Monte Carlo simulations and allow us to have a Monte Carlo algorithm that is ergodic and in principle creates the correct Boltzmann distribution for (44), corresponding to action (41). Details will be provided in other chapters of this section of the handbook; see - Chaps. 78, "Spectral Observables and Gauge Field Couplings in Causal Dynamical Triangulations" and 82, "Semiclassical and Continuum Limits of Four-Dimensional CDT". However, an interesting aspect in four dimensions is that N_4 cannot be kept fixed in these Pachner moves, and it is even impossible in principle to calculate the highest $\widetilde{N}_4(T(N_4), T(N_4'))$ of an intermediate triangulation $T(\widetilde{N}_4)$ that one meets when moving from the triangulation $T_4(N_4)$ to $T_4(N_4')$ by successive application of the Pachner moves [6]. It is unclear what this implies for the practical ergodicity of the used Monte Carlo simulations. It does not necessarily imply that $\widetilde{N}_4$ is very large, but we cannot in principle provide a general expression for $\widetilde{N}_4(T(N_4), T(N_4'))$.

给定两个三角剖分 $T(N_4)$ 和 $T(N_4')$ ，三角剖分 T_4 中存在局部改变 (即所谓的帕克纳移动)，将其应用有限次后，可将 $T(N_4)$ 转变为 $T(N_4')$ 。这些帕克纳移动 [62] 被用于蒙特卡洛模拟中，能够帮助我们得到具有各态历经性的蒙特卡洛算法，该算法原则上可以为式 (44) 生成对应于作用量 (41) 的正确玻尔兹曼分布。具体细节将在本册手章节的其他章节给出；参见第 78 章"因果动态三角剖分中的谱可观测量与规范场耦合"和第 82 章"四维 CDT 的半经典与连续极限"。然而，四维情况下一个值得注意的特点是，N_4 无法在这些帕克纳移动中保持固定，且原则上甚至无法计算: 当我们通过连续应用帕克纳移动从三角剖分 $T_4(N_4)$ 变换到 $T_4(N_4')$ 时，途经的中间三角剖分 $T(\widetilde{N}_4)$ 的最高阶 $\widetilde{N}_4(T(N_4), T(N_4'))$ [6]。目前尚不清楚这对所用蒙特卡洛模拟的实际各态历经性意味着什么。这并不意味着 $\widetilde{N}_4$ 一定很大，但我们原则上无法给出 $\widetilde{N}_4(T(N_4), T(N_4'))$ 的通用表达式。

In the following we will assume that the Monte Carlo simulations work fine, despite the potential problems mentioned above. In the region of coupling-constant space $(\kappa_2, \kappa_4), \kappa_4 > \kappa_4^c(\kappa_2)$ where the partition function $Z(\kappa_2, \kappa_4)$ is well-defined, there is thus no problem with the Euclidean action being unbounded from below. One easily shows, using the so-called Dehn-Sommerville relations, that for a given triangulation T [18, 19] ,

下文我们将假设，尽管存在上述潜在问题，蒙特卡洛模拟仍然可以正常工作。在配分函数 $Z(\kappa_2, \kappa_4)$ 良定义的耦合常数空间区域 $(\kappa_2, \kappa_4), \kappa_4 > \kappa_4^c(\kappa_2)$ 中，欧几里得作用量不存在下无界的问题。利用所谓的德恩-索默维尔关系，我们很容易证明，对于给定的三角剖分 T [18, 19]，

$$2N_4(T) \leq N_2(T) . \tag{49}$$

It is therefore easy to find regions in the coupling-constant space where $S_T(\kappa_2, \kappa_4)$ given by (41) is less than zero and unbounded from below when $N_2(T), N_4(T)$ goes to infinity. However, in this region of coupling-constant space $Z(\kappa_2, \kappa_4)$ is not well-defined and it is not the region considered in EDT. In fact, in the region of large κ_2, which seems most prone to a negative action, the limit $\kappa_4 \to \kappa_4^c(\kappa_2)$ from above is well understood [4] and corresponds to a continuum theory of fractal geometries known as random continuum trees or branched polymers. This continuum limit does not resemble our present universe. The fractal dimension, the Hausdorff dimension, is two on all scales. The same continuum limit continues with decreasing κ_2 until one reaches a critical point κ_2^c, where there is a phase transition, such that for $\kappa_2 < \kappa_2^c$ we encounter a different kind of geometry when $\kappa_4 \to \kappa_4^c(\kappa_2)$. It is a "crumpled" geometry with infinite Hausdorff dimension [4], where a significant fraction of the four-simplices share a single link and the order of the corresponding two vertices is very high [47]. In particular, this seems to be the entropically preferred type of triangulation if no curvature term is present in the action, i.e., $\kappa_2 = 0$. Such highly inhomogeneous triangulations also seem unsuited to describe any theory of quantum gravity.

因此我们很容易在耦合常数空间中找到这样的区域: 当 $N_2(T), N_4(T)$ 趋于无穷时，式 (41) 给出的 $S_T(\kappa_2, \kappa_4)$ 小于零且下无界。但在耦合常数空间的该区域中，$Z(\kappa_2, \kappa_4)$ 没有良好定义，它也不是 EDT 所研究的区域。实际上，在看似最容易出现负作用的大 κ_2 区域中，来自上方的极限 $\kappa_4 \to \kappa_4^c(\kappa_2)$ 是已经被充分理解的 [4]，它对应分形几何的连续统理论，即随机连续树或支化聚合物。该连续统极限并不符合我们当前的宇宙，其分形维数也就是豪斯多夫维数，在所有尺度上都为 2。相同的连续统极限会随着 κ_2 减小一直延续，直到我们到达临界点 κ_2^c，此处会发生相变，因此对于 $\kappa_2 < \kappa_2^c$，当满足 $\kappa_4 \to \kappa_4^c(\kappa_2)$ 时我们会得到一种完全不同的几何。这是一种具有无限豪斯多夫维数的“皱缩”几何 [4]，其中很大一部分四维单形共享同一条链接，对应两个顶点的阶数非常高 [47]。尤其，当作用量中不存在曲率项即 $\kappa_2 = 0$ 时，这种三角化似乎是熵偏好的类型。这种高度非均匀的三角化同样不适合描述任何量子引力理论。

This leaves us with κ_2^c as the only point where one might be able to obtain an interesting theory of quantum gravity. Potentially, this is a good scenario: at the phase transition point the typical geometries one would encounter for $N_4 \to \infty$ could be geometries with a Hausdorff dimension between $d_h = 2$ for $\kappa_2 > \kappa_2^c$ and the $d_h = \infty$ for $\kappa_2 < \kappa_2^c$. If the phase transition was a second-order transition, this scenario could be reasonable, since then one might hope for a smooth transition between the two extreme limits, $d_h = 2$ and $d_h = \infty$. Unfortunately, the Monte Carlo simulations show that the transition is a first-order transition and the geometry at the transition point seems not to be a smooth interpolation between the two types of geometry [41].

这就只剩下 κ_2^c 是唯一可能得到有意义的量子引力理论的位置。这有可能是一个不错的情景: 在相变点，对于 $N_4 \to \infty$，我们得到的典型几何的豪斯多夫维数会介于 $\kappa_2 > \kappa_2^c$ 对应的 $d_h = 2$ 和 $\kappa_2 < \kappa_2^c$ 对应的 $d_h = \infty$ 之间。如果该相变是二级相变，这个情景就是合理的，因为我们可以期待两个极端极限 $d_h = 2$ 和 $d_h = \infty$ 之间存在平滑过渡。但不幸的是，蒙特卡洛模拟表明该相变是一级相变，且相变点的几何似乎并不是两类几何之间的平滑插值 [41]。

This situation does not necessarily imply that an interesting continuum limit cannot be found, but in a Wilsonian context it implies that we have to use a more general action than (41). Adding a suitable term, which could be a measure term or some higher curvature term, in this now three-dimensional coupling constant space, the critical point κ_2^c will turn into a critical line. If the critical line ends, the endpoint would be a candidate for a second-order transition point. Also new phases might appear, and in such a more complicated landscape, there might be a different second order (see [42] for the most recent results).

这种情况并不一定意味着无法得到有意义的连续统极限，但在威尔逊的框架下，这说明我们需要使用比式 (41) 更一般的作用量。在这个三维耦合常数空间中加入一个合适的项 (可以是测度项或某种高阶曲率项) 后，临界点 κ_2^c 会转变为一条临界线。如果临界线有端点，该端点就会是二级相变点的候选。还可能出现新的相，在这种更复杂的图景中，或许存在不同的二级相变 (最新结果参见文献 [42])。

Four-Dimensional CDT

四维因果动态三角剖分 (CDT)

Four-dimensional causal dynamical triangulation (CDT) is a generalization of the simplest version of the two-dimensional CDT described above. The aim is to perform the path integral over geometries on a manifold $[0,1] \times \sum$, where $[0,1]$ denotes a time interval and $\sum$ is a three-dimensional spatial manifold. We discretize the time, the discrete times labelled by t_k . At each discretized time t_k we have a spatial manifold $\sum$, on which we apply the EDT formalism and assign a piecewise linear geometry constructed by gluing together three-dimensional simplices (tetrahedra) with link lengths a , such that the topology of the three-dimensional triangulation T^3 (where the superscript " 3 " means that the triangulation is three-dimensional, not that it is a three-torus) matches that of the manifold $\sum$. Since the geodesic distances on T^3 are uniquely determined, so is the (piecewise linear) geometry. Given a three-dimensional triangulation T_k^3 at time t_k and a three-dimensional triangulation T_{k+1}^3 at time t_{k+1} , we connect these by four-simplices, such that we obtain a four-dimensional triangulation with boundaries T_k^3 and T_{k+1}^3 and such that the topology of this four-dimensional triangulation is $[0,1] \times \sum$. The four-dimensional simplices filling out the "slab" between T_k^3 and T_{k+1}^3 can be of four kinds, depending on how many vertices the four-dimensional simplices share with a tetrahedron belonging to T_k^3 . If the four-simplex shares four vertices with T_k^3 , i.e., is a tetrahedron in T_k^3 , and thus one vertex with the triangulation T_{k+1}^3 , we denote it a $T^{(4,1)}$ simplex. If it shares three vertices with a tetrahedron in T_k^3 , i.e., forms a triangle in T_k^3 , and two vertices with T_{k+1}^3 , i.e., forms a link in T_{k+1}^3 , we denote it a $T^{(3,2)}$ simplex. The four-simplices $T^{(2,3)}$ and $T^{(1,4)}$ are defined similarly. The whole construction is clearly a generalization of the construction of two-dimensional CDT, where in an analogous notation we would have two kinds of two-simplices, $T^{(2,1)}$ and $T^{(1,2)}$. The links of the four-dimensional simplices which are also links in T_k^3 and T_{k+1}^3 are assigned a positive length a , while the links connecting vertices in T_k^3 to vertices in T_{k+1}^3 are viewed as time-like, i.e., we write, in analogy with (28),

四维因果动态三角剖分 (CDT) 是上述二维 CDT 最简形式的推广。其目标是对流形 $[0,1]\times\sum$ 上的几何进行路径积分，其中 $[0,1]$ 表示时间间隔，$\sum$ 是三维空间流形。我们对时间进行离散化，离散时间由 t_k 标记。在每个离散时间 t_k ，我们都有一个空间流形 $\sum$ ，我们对其应用 EDT 形式，并赋予它一个由链接长度为 a 的三维单纯形 (四面体) 粘合而成的分段线性几何，使得三维三角剖分 T^3 (上标“3”表示该三角剖分是三维的，而非指它是三维环面) 的拓扑与流形 $\sum$ 的拓扑匹配。由于 T^3 上的测地线距离是唯一确定的，因此其 (分段线性) 几何也是唯一确定的。已知时间 t_k 处的三维三角剖分 T_k^3 和时间 t_{k+1} 处的三维三角剖分 T_{k+1}^3 ，我们用四单纯形将二者连接起来，从而得到一个以 T_k^3 和 T_{k+1}^3 为边界的四维三角剖分，且该四维三角剖分的拓扑为 $[0,1]\times\sum$ 。填充在 T_k^3 和 T_{k+1}^3 之间“层”的四维单纯形可分为四种，分类依据是该四维单纯形与属于 T_k^3 的四面体共享多少个顶点。如果四单纯形与 T_k^3 共享四个顶点，即它本身是 T_k^3 中的一个四面体，因此仅与 T_{k+1}^3 中的三角剖分共享一个顶点，我们将其记为 $T^{(4,1)}$ 单纯形。如果它与 T_k^3 中的一个四面体共享三个顶点，即它在 T_k^3 中构成一个三角形，且与 T_{k+1}^3 共享两个顶点，即它在 T_{k+1}^3 中构成一条链接，我们将其记为 $T^{(3,2)}$ 单纯形。$T^{(2,3)}$ 单纯形和 $T^{(1,4)}$ 单纯形可按类似方式定义。整个构造显然是二维 CDT 构造的推广，在类似的标记下，二维 CDT 会有两种二单纯形: $T^{(2,1)}$ 和 $T^{(1,2)}$ 。同时属于 T_k^3 和 T_{k+1}^3 的四单纯形链接被赋予正长度 a ，而连接 T_k^3 顶点与 T_{k+1}^3 顶点的链接被视为类时链接，也就是说，仿照式 (28)，我们写出

$$a_t^2=-\alpha a^2,\ \alpha>0. \tag{50}$$

A complete triangulation of the manifold $[0,1]\times\sum$ is now obtained by repeating the above procedure for $k=1,2,\ldots,s$, yielding a four-dimensional triangulation with spatial boundaries T_1^3 and T_s^3 and spatial slices $T_k^3, 1<k<s$. The corresponding Regge action for such a geometry is still very simple, although slightly more complicated than (41), since we have introduced a parameter α , which will allow us to perform a rotation of the geometry with Lorentzian signature to one with Euclidean signature. The Lorentzian action for such a Lorentzian triangulation T_{lor} , expressed using the notation from (41), can be written as [20, 31]

现在通过对 $k=1,2,\ldots,s$ 重复上述步骤，得到流形 $[0,1]\times\sum$ 的完整三角剖分，最终得到一个带有空间边界 T_1^3 、T_s^3 和空间切片 $T_k^3, 1<k<s$ 的四维三角剖分。这类几何对应的 Regge 作用量仍然非常简单，仅比式 (41) 稍复杂——这是因为我们引入了参数 α ，通过该参数我们可以将洛伦兹号差的几何转动为欧几里得号差的几何。利用式 (41) 的符号，这类洛伦兹三角剖分 T_{lor} 对应的洛伦兹作用量可写为 [20, 31]

$$\begin{gathered}
S_{T_{\text{lor}}}=\kappa_2\sqrt{4\alpha+1}\left[\frac{\pi}{2}N_2^{\text{TL}}+\right. \\
N_4^{(4,1)}\left(-\frac{\sqrt{3}}{\sqrt{4\alpha+1}}\operatorname{arcsinh}\frac{1}{2\sqrt{2}\sqrt{3\alpha+1}}-\frac{3}{2}\arccos\frac{2\alpha+1}{2(3\alpha+1)}\right)+ \\
N_4^{(3,2)}\left(\frac{\sqrt{3}}{4\sqrt{4\alpha+1}}\operatorname{arcsinh}\frac{\sqrt{3}\sqrt{12\alpha+7}}{2(3\alpha+1)}-\right. \\
\left.\left.\frac{3}{4}\left(2\arccos\frac{-1}{2\sqrt{2}\sqrt{2\alpha+1}\sqrt{3\alpha+1}}+\arccos\frac{4\alpha+3}{4(2\alpha+1)}\right)\right)\right] \\
-\kappa_4\left(N_4^{(4,1)}\frac{\sqrt{8\alpha+3}}{96}+N_4^{(3,2)}\frac{\sqrt{12\alpha+7}}{96}\right).
\end{gathered} \tag{51}$$

$N_4^{(4,1)}$ and $N_4^{(3,2)}$ denotes the total number of four-simplices of types $T^{(4,1)}$ and $T^{(1,4)}$ and of types $T^{(3,2)}$ and $T^{(2,3)}$, respectively, in the triangulation T_{lor} .N_2^{TL} denotes the number of time-like triangles in the triangulation, i.e., triangles with one spacelike link and two time-like links. Of course κ_2 also multiplies the number of N_2^{SL} of spacelike triangles, but this number has been expressed in terms of the numbers $N_4^{(4,1)}$ and $N_4^{(3,2)}$ of four-simplices, by virtue of the special time-slicing structure present for a CDT triangulation. Finally, we have ignored a Regge boundary action term, coming from the two boundaries, since in the actual computer simulations we replace the manifold $[0,1]\times\sum$ with the manifold $S^1\times\sum$. As we will see, the setup of the computer simulations will be such that in most cases there will be no difference between choosing $[0,1]$ or S^1 . The action is written in a way that makes it real for all positive α and purely imaginary for $\alpha<-7/12$. Of course our starting point is a Lorentzian geometry with $\alpha>0$, but now, like in the two-dimensional case, we can make a rotation to Euclidean geometry by performing a rotation $\alpha\to-\alpha$ in the lower complex α -plane, assuming $\alpha>7/12$. One then obtains the action

$N_4^{(4,1)}$ 和 $N_4^{(3,2)}$ 分别表示三角剖分中类型为 $T^{(4,1)}$ 、$T^{(1,4)}$ 和类型为 $T^{(3,2)}$ 、$T^{(2,3)}$ 的四维单形总数；T_{lor} .N_2^{TL} 表示三角剖分中的类时三角形数量，即具有一条类空链接和两条类时链接的三角形。当然 κ_2 同样会乘以类空三角形的数量 N_2^{SL} ，但由于 CDT 三角剖分具有特殊时间分片结构，该数量已经可以用四维单形数量 $N_4^{(4,1)}$ 和 $N_4^{(3,2)}$ 表示。最后我们忽略了来自两个边界的 Regge 边界作用项，因为在实际计算机模拟中我们用流形 $S^1\times\sum$ 替代了流形 $[0,1]\times\sum$ 。正如我们将会看到的，计算机模拟的设置使得大多数情况下选择 $[0,1]$ 或 S^1 没有区别。该作用量的构造满足: 对所有正的 α 它是实的，对 $\alpha<-7/12$ 它是纯虚数。当然我们的起点是满足 $\alpha>0$ 的洛伦兹几何，但是和二维情况一样，在假设 $\alpha>7/12$ 成立的条件下，我们可以通过在 α 复平面下半部分做转动 $\alpha\to-\alpha$ ，将其转动为欧几里得几何，随后便可得到作用量

$$
\begin{aligned}
S_{T_{\text{eucl}}} = -\kappa_2\sqrt{4\tilde{\alpha}-1}\Bigg[\pi\left(N_0-\chi+\frac{1}{2}N_4^{(4,1)}+N_4^{(3,2)}\right)+ \\
N_4^{(4,1)}\left(-\frac{\sqrt{3}}{\sqrt{4\tilde{\alpha}-1}}\arcsin\frac{1}{2\sqrt{2}\sqrt{3\tilde{\alpha}-1}}+\frac{3}{2}\arccos\frac{2\tilde{\alpha}-1}{6\tilde{\alpha}-2}\right)+ \\
N_4^{(3,2)}\left(+\frac{\sqrt{3}}{4\sqrt{4\tilde{\alpha}-1}}\arccos\frac{6\tilde{\alpha}-5}{6\tilde{\alpha}-2}+\frac{3}{4}\arccos\frac{4\tilde{\alpha}-3}{8\tilde{\alpha}-4}+\right. \\
\left.\frac{3}{2}\arccos\frac{1}{2\sqrt{2}\sqrt{2\tilde{\alpha}-1}\sqrt{3\tilde{\alpha}-1}}\right]+\kappa_4\left(N_4^{(3,2)}\frac{\sqrt{12\tilde{\alpha}-7}}{96}+N_4^{(4,1)}\frac{\sqrt{8\tilde{\alpha}-3}}{96}\right)\Bigg].
\end{aligned}
\tag{52}
$$

Analogous to the two-dimensional case, we have

与二维情况类似，我们有

$$
S_{T_{\text{rmlor}}}(\kappa_2,\kappa_4,\alpha)\to S_{T_{\text{lor}}}(\kappa_2,\kappa_4,-\alpha)=iS_{T_{\text{eucl}}}(\kappa_2,\kappa_4,\tilde{\alpha}),\ \alpha=\tilde{\alpha}>\frac{7}{12}.
$$

(53)

In the same way as the constraint $\tilde{\alpha}>1/4$ in the two-dimensional case was linked to the triangle inequality (and still is in (52)), the inequality $\tilde{\alpha}>7/12$ is linked to the geometry of a $T^{(3,2)}$ simplex: for $\tilde{\alpha}=7/12$ the "time-like" distance between the opposing spatial link and spatial triangle in the simplex becomes zero. In (52) we have replaced N_2^{TL} by $N_0-\chi+\frac{1}{2}N_4^{(4,1)}+N_4^{(3,2)}$, where N_0 denotes the number of vertices in the triangulation T and χ the Euler characteristic of the manifold. This relation again follows from the Dehn-Sommerville relations for four-dimensional CDT triangulations. One can check that for $\tilde{\alpha} = 1$, one precisely recovers the EDT expression (43).

正如二维情况下约束条件 $\tilde{\alpha} > 1/4$ 与三角不等式相关 (在式 (52) 中依然如此)，不等式 $\tilde{\alpha} > 7/12$ 与 $T^{(3,2)}$ 单形的几何性质相关: 对于 $\tilde{\alpha} = 7/12$ ，单形中对向空间棱与空间三角形之间的类时距离变为零。在式 (52) 中我们将 N_2^{TL} 替换为 $N_0 - \chi + \frac{1}{2}N_4^{(4,1)} + N_4^{(3,2)}$ ，其中 N_0 表示三角剖分 T 的顶点数，χ 是流形的欧拉示性数。该关系可由四维 CDT 三角剖分的德恩-索默维尔关系导出，也可验证对于 $\tilde{\alpha} = 1$ ，我们可以恰好得到 EDT 的表达式 (43)。

In the Monte Carlo simulations, using (generalized (We have to use slightly generalized Pachner moves to preserve the CDT foliation structure [31].)) Pachner moves to change the triangulations, the topology of the triangulations is kept fixed and we can ignore the χ -term. We will do that in the following.

在蒙特卡洛模拟中，我们使用 (广义化的 (我们需要使用略微广义化的帕赫纳移动来保留 CDT 的叶状结构 [31]。)) 帕赫纳移动改变三角剖分，过程中三角剖分的拓扑保持固定，因此我们可以忽略 χ 项，下文将沿用这一处理。

Let us again stress that while an expression like (52) looks somewhat complicated because of the $\tilde{\alpha}$ -dependence, it is, like (43), exceedingly simple, since the action of a triangulation T just depends on the three global numbers $N_0(T), N_4^{(4,1)}(T)$, and $N_4^{(3,2)}(T)$. Again the partition function for CDT quantum gravity is simply the generating function for the number of triangulations with given $N_0, N_4^{(4,1)}$, and $N_4^{(3,2)}$, with suitable indeterminates, now depending not only on κ_2 and κ_4 , but also on $\tilde{\alpha}$. By redefining the coupling constants we can make this simplicity explicit by writing

我们需要再次强调，尽管类似式 (52) 的表达式因依赖 $\tilde{\alpha}$ 看上去稍显复杂，但它和式 (43) 一样极为简洁: 三角剖分 T 的作用量仅依赖三个全局量 $N_0(T), N_4^{(4,1)}(T)$ 和 $N_4^{(3,2)}(T)$ 。同样，CDT 量子引力的配分函数就是给定 $N_0, N_4^{(4,1)}$ 和 $N_4^{(3,2)}$ 的三角剖分数量的生成函数，配分函数带有合适的不定元，它不仅依赖 κ_2 和 κ_4 ，还依赖 $\tilde{\alpha}$ 。通过重新定义耦合常数，我们可以将这种简洁性明确写为

$$S_T(k_0, k_4, \Delta) = -(k_0 + 6\Delta) N_0(T) + k_4 \left(N_4^{(4,1)}(T) + N_4^{(3,2)}(T) \right) + \Delta N_4^{(4,1)}(T).$$

(54)

This action is still formally equal to the Regge version of the Einstein-Hilbert action for a piecewise linear manifold constructed as described above, where the spatial links have length a and the "time-like" links a length $\sqrt{\tilde{\alpha}}a$. It is still true that $k_0 \propto a^2/G$, while Δ is a rather complicated function of k_0, k_4 , and $\tilde{\alpha}$ such that $\Delta = 0$ corresponds to $\tilde{\alpha} = 1$. However, from the computer simulations to be discussed below, it will be clear we cannot maintain such an interpretation. It is thus a more fruitful, Wilsonian interpretation of (54) to say that our starting point is the Regge action with

该作用量形式上仍然等于前文所述分段线性流形的爱因斯坦-希尔伯特作用量的里奇版本，其中空间棱的长度为 a ，类时棱的长度为 $\sqrt{\tilde{\alpha}}a$ 。仍然满足 $k_0 \propto a^2/G$ ，同时 Δ 是 k_0, k_4 和 $\tilde{\alpha}$ 的相当复杂的函数，使得 $\Delta = 0$ 对应 $\tilde{\alpha} = 1$ 。但从下文将要讨论的计算机模拟结果可知，我们无法维持这一解释。因此对式 (54) 更富有成效的威尔逊式解读是: 我们的起点是带有如下条件的里奇作用量

$$\alpha = \tilde{\alpha} = 1 \text{ , and } \Delta \text{ is an independent coupling constant.} \tag{55}$$

In this way $\Delta = 0$ will correspond to the Euclidean Einstein-Hilbert action (43), but where the geometries have a time foliation coming from the Lorentzian geometries described above, and the new coupling constant Δ is a Wilsonian enlargement of the coupling-constant space from (k_0, k_4) to (k_0, k_4, Δ) . We will explore this space in the search for potentially interesting phase transitions of the lattice system, which could be associated with a UV fixed point for quantum gravity (In this sense the situation becomes similar to the EDT situation, where one has to enlarge the (k_0, k_4) coupling-constant space defined in (43) by some new coupling constant in order to obtain an interesting result, as already mentioned.). In this context let us mention that we can of course rewrite (54) as

通过这种方式，$\Delta = 0$ 将对应欧几里得爱因斯坦-希尔伯特作用量 (43)，不过此处几何具有来自上文所述洛伦兹几何的时间叶状结构，且新耦合常数 Δ 是威尔逊意义上对耦合常数空间的扩充，即将耦合空间从 (k_0, k_4) 扩展到 (k_0, k_4, Δ) 。我们将探究该空间，寻找格点系统中可能存在的有意义相变，这类相变可能与量子引力的紫外不动点相关 (从这个意义上说，该情况与 EDT 的情况类似，如前文所述，在 EDT 中人们必须通过引入新耦合常数来扩充 (43) 中定义的 (k_0, k_4) 耦合常数空间，才能得到有价值的结果)。在此语境下我们需要指出，我们当然可以将 (54) 重写为

$$S_T\left(\tilde{k}_0, k_{4,1}, k_{32}\right) = -\tilde{k}_0 N_0(T) + k_{41} N_4^{(4,1)}(T) + k_{32} N_4^{(3,2)}(T) \tag{56}$$

to emphasize that the action is the most general action that only depends linearly on the global number of simplices or subsimplices in a CDT triangulation (See [31] for a classification of time- and spacelike (sub)simplices of a CDT configuration and the constraints these numbers satisfy. There are 10 different types of (sub)simplices and 7 constraints.).

以此强调，该作用量是仅线性依赖于 CDT 三角剖分中单纯形或子单纯形总个数的最广义作用量 (关于 CDT 构型中类时和类空 (子) 单纯形的分类以及这些个数满足的约束，参见 [31]，其中共有 10 种不同类型的 (子) 单纯形和 7 个约束条件)。

To summarize, our four-dimensional CDT partition function is

总而言之，我们的四维 CDT 配分函数为

$$Z(k_0, k_4, \Delta) = \sum_T \frac{1}{C_T} \mathrm{e}^{-S_T(k_0, k_4, \Delta)} \tag{57}$$

$$= \sum_{N_0, N_4^{(4,1)}, N_4^{(3,2)}} \mathrm{e}^{\left[(k_0+6\Delta)N_0 - k_4\left(N_4^{(4,1)}+N_4^{(3,2)}\right) - \Delta N_4^{(4,1)}\right]} \mathcal{N}\left(N_0, N_4^{(4,1)}, N_4^{(3,2)}\right),$$

where the summation is over CDT triangulations and $\mathcal{N}\left(N_0, N_4^{(4,1)}, N_4^{(3,2)}\right)$ denotes the number of such triangulations with N_0 vertices, $N_4^{(4,1)}$ simplices of type $T^{(4,1)}$ plus type $T^{(1,4)}$, and $N_4^{(3,2)}$ simplices of type $T^{(3,2)}$ plus type $T^{(2,3)}$. We now turn to the discussion of the phase diagram of this statistical system.

其中求和遍历所有 CDT 三角剖分，$\mathcal{N}\left(N_0, N_4^{(4,1)}, N_4^{(3,2)}\right)$ 表示这类三角剖分的数目: 这些三角剖分具有 N_0 个顶点，$N_4^{(4,1)}$ 个 $T^{(4,1)}$ 型加 $T^{(1,4)}$ 型单纯形，以及 $N_4^{(3,2)}$ 个 $T^{(3,2)}$ 型加 $T^{(2,3)}$ 型单纯形。下面我们来讨论这个统计系统的相图。

Search for a UV Fixed Point in CDT

在 CDT 中搜索紫外不动点

The enlargement of the CDT coupling-constant space with the coupling constant Δ leads to an amazingly complex phase diagram (The phase diagram presented in the first articles [21,29-31] was simpler since it missed the C_b phase, discovered in [35, 36, 39] .) [38] shown in Fig. 3. It shows the (k_0, Δ) coupling constant plane. As discussed above, it is impossible to keep $N_4 = N_4^{(4,1)} + N_4^{(3,2)}$ fixed in the Monte Carlo simulations. However, they can be conducted in such a way that measurements of the observables used to identify the phase transitions are performed for a given chosen value of N_4 . In this way k_4 does not enter actively as a coupling constant influencing the observables (In many of the simulations it has been more convenient to instead keep $N_4^{(4,1)}$ fixed at the measurements.). This is why the figure only shows the (k_0, Δ) coupling-constant plane. The physics related to the coupling-constant k_4 can be recovered by performing measurements for many different values of N_4 . Explicitly we have

引入耦合常数 Δ 扩大 CDT 耦合常数空间后，得到了极为复杂的相图 (发表于 [21,29-31] 的初代相图更简单，因为其中遗漏了在 [35, 36, 39] 中发现的 C_b 相)[38]，如图 3 所示。图中展示的是 (k_0, Δ) 耦合常数平面。正如前文所述，蒙特卡洛模拟中无法固定 $N_4 = N_4^{(4,1)} + N_4^{(3,2)}$ 。但我们可以调整模拟方式，针对选定的 N_4 值测量用于识别相变的观测量。通过这种方式，k_4 不会作为耦合常数主动影响观测量 (许多模拟中更方便的做法是在测量时固定 $N_4^{(4,1)}$)。因此该图仅展示了 (k_0, Δ) 耦合常数平面。我们可以通过对多个不同的 N_4 取值分别测量，还原和耦合常数 k_4 相关的物理性质。具体形式如下

$$Z(k_0, \Delta, k_4) = \sum_{N_4} \mathrm{e}^{-k_4 N_4} Z_{N_4}(k_0, \Delta), \tag{58}$$

$$Z_{N_4}(k_0, \Delta) = \sum_{T(N_4)} \frac{1}{C_T} \mathrm{e}^{(k_0+\Delta) N_0(T(N_4)) - \Delta N_4^{(4,1)}(T(N_4))}, \tag{59}$$

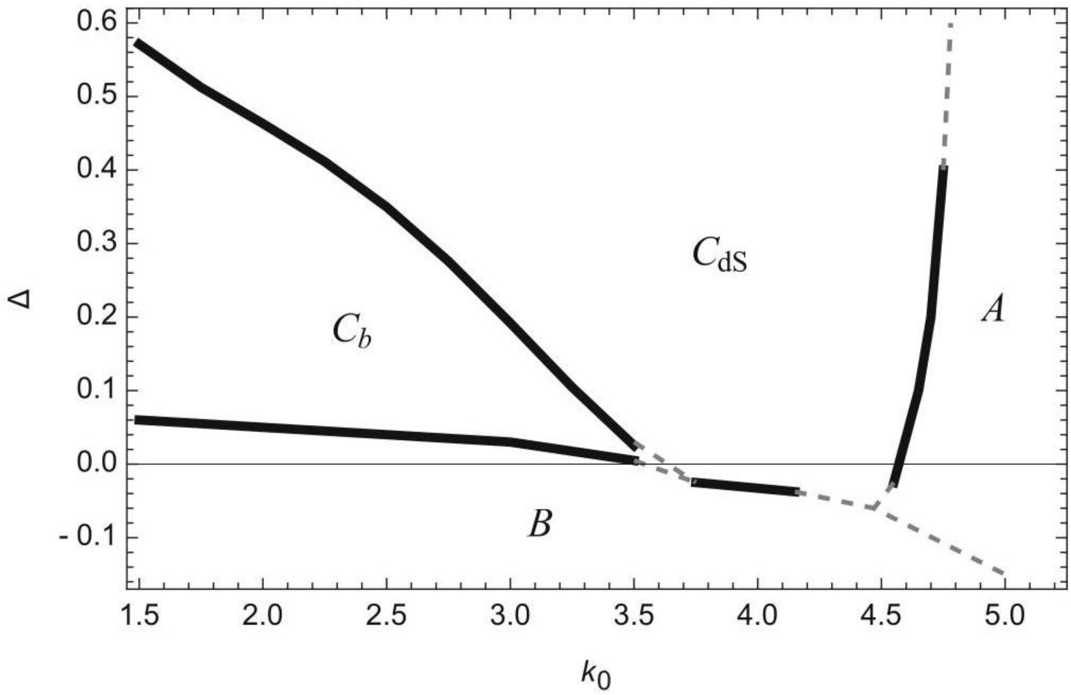

Fig. 3 The CDT phase diagram. Phase transition between phase C_{dS} and C_b is second order when the topology of a spatial slice is S^3 , as is the transition between C_b and B . The transition between C_{dS} and A and the transition between A and B are first-order transitions. The transition between C_{dS} and B is still under investigation

图 3 CDT 相图。当空间切片的拓扑为 S^3 时，C_{dS} 相与 C_b 相之间的相变是二级相变，C_b 相与 B 相之间的相变同样为二级相变。C_{dS} 相与 A 相之间、A 相与 B 相之间的相变均为一级相变。C_{dS} 相与 B 相之间的相变仍在研究中

where the summation is over all triangulations $T(N_4)$ with a fixed number N_4 of four-simplices.

其中求和遍历所有满足固定四单纯形数量为 N_4 的三角剖分 $T(N_4)$

Figure 3 shows the phase transition lines between the various phases, denoted A , B, C_b , and C_{dS} . Here the subscript dS stands for ”de Sitter” and b for ”bifurcation.” Phase A and phase B are most likely not relevant for a four-dimensional quantum gravity theory, and the phase transitions between phases A and C_{dS} , as well as between phases A and B , are first-order transitions. In the case where $\sum$ is a three-dimensional manifold with the topology of S^3 , there is good numerical evidence that the phase transition between B and C_b , as well as the transition between C_b and C_{dS} , are second-order transitions (The order of the transition between phase C_{dS} and phase B is most likely also a higher-order transition, but it is not entirely settled yet.). The A phase can be viewed as the CDT version of the branched-polymer phase of EDT and the B phase as the CDT version of the crumpled phase. The details of these phases are discussed in Chap. 82, ”Semiclassical and Continuum Limits of Four-Dimensional CDT” in this section of the handbook. Here we concentrate on discussing the phase transition between phase C_b and C_{dS} , since this transition has a relatively transparent physical origin. There is good numerical evidence that the universe ”observed” in phase C_{dS} can be considered as homogeneous and isotropic in the spatial directions, while the homogeneity is broken in phase C_b . This symmetry breaking appears to happen smoothly (in accordance with the higher-order nature of the transition)

[35, 38] , such that regions of inhomogeneity become more and more pronounced the deeper we move into phase C_b , starting from the border between phase C_b and phase C_{dS} . It is tempting to conjecture that the seeds of inhomogeneity of geometry at the phase transition line could act as seeds for matter inhomogeneity, in case matter was added to the model and maybe in this way be important for the first galaxy formations. In particular this would present an intriguing scenario if the phase transition line could be associated with a UV fixed point. First of all, as already mentioned, a UV fixed point in a theory of gravity is central to the asymptotic safety scenario, and finding it in our lattice approach is central to the idea that one can use the lattice theory as a non-perturbative definition of quantum gravity. Second, it is also often assumed that the ”origin” of the universe is associated with the theory of gravity at short distances (some kind of big bang scenario), and this should then naturally relate to physics close to the UV fixed point. Inhomogeneity as part of this UV physics could then be important for the formation of structure in a universe with matter.

图 3 展示了不同相之间的相变线，分别记为 A 、B, C_b 和 C_{dS} 。此处下标 dS 代表“德西特”，b 代表“分岔”。A 相和 B 相与四维量子引力理论几乎无关，且 A 相与 C_{dS} 相之间、A 相与 B 相之间的相变均为一级相变。当 $\sum$ 为具有 S^3 拓扑的三维流形时，有充分数值证据表明，B 相与 C_b 之间以及 C_b 相与 C_{dS} 之间的相变均为二级相变 (C_{dS} 相与 B 相之间的相变阶数很可能也是高阶相变，但目前尚未完全确定)。A 相可视为因果动力三角剖分 (CDT) 中对应欧几里得动力三角剖分 (EDT) 支化聚合物相的版本，B 相则是 CDT 中对应皱缩相的版本。这些相的细节已在本手册该部分的第 82 章“四维 CDT 的半经典与连续极限”中讨论。本文我们集中讨论 C_b 相与 C_{dS} 相之间的相变，因为该相变的物理起源相对清晰。有充分数值证据表明，在 C_{dS} 相中“观测到”的宇宙在空间方向上可认为是均匀各向同性的，而均匀性在 C_b 相中被破坏。这种对称性破缺似乎是平稳发生的 (符合该相变的高阶性质 [35, 38]): 从 C_b 相与 C_{dS} 相的边界出发，越深入 C_b 相内部，非均匀区域就越显著。我们很容易做出猜想: 若在模型中加入物质，相变线处几何非均匀性的“种子”可以充当物质非均匀性的种子，这或许会对早期星系形成起到重要作用。如果这条相变线确实与紫外不动点相关，那么这就会构成一个尤其引人入胜的图景。首先，正如前文所述，引力理论中的紫外不动点是渐近安全情景的核心，在我们的格点方法中找到它，是“格点理论可作为量子引力的非微扰定义”这一观点的核心。其次，人们通常也认为宇宙的“起源”与短距离下的引力理论相关 (某种大爆炸情景)，因此这自然会与紫外不动点附近的物理联系起来。作为这类紫外物理一部分的非均匀性，对于含物质宇宙中结构的形成可能十分重要。

Therefore, confronted with a second-order phase transition line, the phase transition line between the C_b and C_{dS} phases, the obvious question of interest is whether or not there is a non-perturbative UV fixed point associated with this line.

因此，面对二级相变线——也就是 C_b 相与 C_{dS} 相之间的相变线，我们自然会提出一个关键问题: 这条线是否关联着一个非微扰紫外不动点。

Search for a UV Fixed Point in a ϕ^4 Theory

在 ϕ^4 理论中寻找紫外不动点

In order to address this question, let us step back and briefly recall how it has been addressed in ordinary ϕ^4 scalar field theory in four-dimensional flat (Euclidean) spacetime (Since the Higgs field ϕ in the standard model is governed by a ϕ^4 field theory (embedded in a larger theory), the existence or nonexistence of such a

UV fixed point is actually important for a standard model.). Consider the scalar ϕ^4 theory defined on a four-dimensional hypercubic lattice with lattice spacing a. We denote the integer lattice coordinates of the vertices by $n = \left(n^1, \ldots, n^4\right)$ and the spacetime coordinates of these lattice points by $x_n = an$. A lattice scalar field ϕ takes values on the lattice vertices and we use the notation $\phi(n)$ or $\phi(x_n)$. The action is

为了解决这个问题，我们退一步，简要回顾一下在四维平直 (欧氏) 时空的普通 ϕ^4 标量场理论中，该问题是如何处理的 (由于标准模型中的希格斯场 ϕ 由 ϕ^4 场论描述 (嵌入更大的理论中)，这类紫外不动点是否存在实际上对标准模型而言十分重要)。考虑定义在格间距为 \$a\$ 的四维超立方格点上的标量 ϕ^4 理论。我们将顶点的整数格点坐标记为 $n = \left(n^1, \ldots, n^4\right)$ ，将这些格点的时空坐标记为 $x_n = an$ 。格点标量场 ϕ 定义在格点顶点上，我们使用记号 $\phi(n)$ 或 $\phi(x_n)$ 。其作用量为

$$S[\phi, \mu, \lambda; a] = \sum_n a^4 \left(\frac{1}{2} \sum_{i=1}^{4} \frac{\left(\phi(n+\hat{i}) - \phi(n)\right)^2}{a^2} + \frac{1}{2} \frac{\mu}{a^2} \phi^2(n) + \frac{1}{4!} \lambda \phi^4(n) \right),$$

(60)where $\hat{i}$ denotes the unit vector in direction i .

其中 $\hat{i}$ 表示方向 i 上的单位向量

The theory has two dimensionless lattice coupling constants μ and λ . In this coupling-constant space there is a phase transition line between a symmetric phase where $\langle \phi(n) \rangle = 0$ and a symmetry-broken phase where $\langle \phi(n) \rangle \neq 0$. The symmetry broken is $\phi(n) \to -\phi(n)$. This transition line is a second-order phase transition line, and the correlation length between the fields at different lattice points diverges when one approaches the transition line. The question is whether this phase transition line be used to define a non-perturbative UV fixed point for the ϕ^4 quantum field theory. The tentative continuum quantum field theory is defined by its two renormalized continuum coupling constants m_R and λ_R , the continuum mass, and the continuum ϕ^4 coupling constant. They can be extracted from the two-point correlator and the four-point correlator. The lattice two-point function is characterized by a lattice correlation length ξ . We can write

该理论有两个无量纲格点耦合常数 μ 和 λ 。在这个耦合常数空间中，对称相 (满足 $\langle \phi(n) \rangle = 0$) 和对称破缺相 (满足 $\langle \phi(n) \rangle \neq 0$) 之间存在一条相变线，破缺的对称性为 $\phi(n) \to -\phi(n)$ 。这条相变线是二级相变线，当趋近相变线时，不同格点上场之间的关联长度发散。问题在于，这条相变线能否用来为 ϕ^4 量子场论定义一个非微扰紫外不动点。暂定的连续量子场论由两个重整化连续耦合常数定义: 连续质量和连续 ϕ^4 耦合常数，即 m_R 和 λ_R 。它们可以从两点关联函数和四点关联函数中提取出来。格点两点函数由格点关联长度 ξ 刻画，我们可以写出

$$\xi(\mu, \lambda) = \lim_{|n-n'| \to \infty} \frac{-\log \langle (\phi(n) - \langle \phi \rangle)(\phi(n') - \langle \phi \rangle) \rangle}{|n - n'|}, \quad m_R = \frac{1}{a\xi}. \tag{61}$$

We assume for simplicity that the coupling constants are chosen such that we are in the symmetric phase, where $\langle \phi \rangle = 0$. Let us not discuss in detail how to define λ_R (for details see, for instance,[61]), but only state that insisting that λ_R is constant defines a path $(\mu(\xi), \lambda(\xi))$ in the lattice coupling-constant space (μ, λ) . For each point on this path we can calculate a correlation length ξ using (61), and we use these ξ as a parametrization of the path. If the path meets the second-order transition line at a point (μ^c, λ^c) , it implies that $\xi \to \infty$ at this point. This point can serve as a UV fixed point, since we now demand that m_R is constant along the path, i.e., the lattice spacing a becomes a function of ξ via (61),

为简化起见，我们假设耦合常数的选取使我们处于对称相，满足 $\langle\phi\rangle = 0$ 。我们不详细讨论如何定义 λ_R (例如，细节可参见文献 [61])，只需说明: 坚持要求 λ_R 为常数即可在格点耦合常数空间 (μ, λ) 中定义一条路径 $(\mu(\xi), \lambda(\xi))$ 。对这条路径上的每个点，我们可以利用 (61) 计算关联长度 ξ ，并将这些 ξ 作为路径的参数化。如果路径在点 (μ^c, λ^c) 处与二级相变线相交，这意味着在该点 $\xi \to \infty$ 。这个点可以用作紫外不动点，因为我们现在要求 m_R 沿路径保持常数，即格间距 a 通过 (61) 成为 ξ 的函数:

$$a(\xi) = \frac{1}{m_R \xi}, \text{ i.e., } a(\xi) \to 0 \text{ for } \xi \to \infty. \tag{62}$$

We conclude that by following a path in the bare, dimensionless coupling-constant space, where continuum observables are kept fixed, one is led to a UV fixed point, provided it exists. If the UV point does not exist, the path will be such that ξ never reaches infinity, no matter where we start in the bare coupling constant space. According to (62) this implies that we cannot remove the UV cutoff a .

我们得出结论: 沿着裸无量纲耦合常数空间中保持连续可观测量固定的路径，只要紫外不动点存在，我们就能得到一个紫外不动点。如果紫外不动点不存在，那么无论我们在裸耦合常数空间中从何处出发，路径都会使得 ξ 永远无法达到无穷大。根据 (62)，这意味着我们无法移除紫外截断 a 。

The approach to the UV fixed point is governed by the β -function (The β -function is a function of λ and μ , but close to the fixed point one can ignore the μ -dependence.), which relates the change in λ to the change in $a(\xi) = 1/(m_R\xi)$ as we move along the trajectory of constant m_R, λ_R ,

趋近紫外固定点的过程由 β 函数主导 (β 函数是 λ 和 μ 的函数，但在接近固定点时可以忽略对 μ 的依赖)，当我们沿常数 m_R, λ_R 的轨迹移动时，该函数将 λ 的变化与 $a(\xi) = 1/(m_R\xi)$ 的变化关联起来，

$$-a \frac{\mathrm{d}\lambda}{\mathrm{d}a}\Big|_{m_R, \lambda_R} = \xi \frac{\mathrm{d}\lambda}{\mathrm{d}\xi}\Big|_{m_R, \lambda_R} = \beta(\lambda). \tag{63}$$

Since $\lambda(\xi)$ stops changing when $\xi \to \infty$, we have $\beta(\lambda^c) = 0$, and expanding the β -function to first order one finds

当 $\xi \to \infty$ 时，$\lambda(\xi)$ 不再变化，因此我们得到 $\beta(\lambda^c) = 0$ ，将 β 函数展开到一阶可得

$$\lambda(\xi) = \lambda^c + \text{ const. } \xi^{\beta'(\lambda^c)}, \ \beta'(\lambda) = \frac{d\beta}{d\lambda}. \tag{64}$$

It follows from (64) that $\beta'(\lambda^c) < 0$ at a UV fixed point.

由式 (64) 可得，在紫外固定点处 $\beta'(\lambda^c) < 0$ 。

The correlation length ξ clearly plays a major role in the above scenario. It will be convenient to replace it with a finite lattice volume by using so-called finite-size scaling. Assume we have a finite hypercubic lattice. The volume is then $V = Na^4$, where N is the number of hypercubes. We keep the ratio between the linear size of the lattice and the correlation length fixed,

关联长度 ξ 显然在上述情景中发挥核心作用。利用所谓有限尺寸标度，我们可以方便地将其替换为有限格点体积。假设我们有有限超立方格点，其体积为 $V = Na^4$ ，其中 N 是超立方体的数量。我们保持格点线度与关联长度的比值固定，

$$\frac{\xi}{N^{1/4}} = \frac{1}{(a(\xi)\, m_R)\, N^{1/4}} = \frac{1}{m_R V^{1/4}}. \tag{65}$$

Thus, if we are moving along a trajectory with constant m_R and λ_R in the bare (μ, λ) -coupling constant plane and change N according to (65), the finite continuum volume stays fixed. Assuming that there is a UV fixed point, such that $a(\xi) \to 0$, we see that N can go to infinity even if V stays finite and that the correlation length ξ in (64) can be substituted by a dependence on the linear size $N^{1/4}$ in lattice units of the spacetime, leading to

因此，如果我们在裸 (μ, λ) 耦合常数平面沿 m_R 和 λ_R 为常数的轨迹移动，并按照式 (65) 改变 N ，有限连续体积会保持不变。若存在一个满足 $a(\xi) \to 0$ 的紫外固定点，我们可以看到，即使 V 保持有限，N 也可以趋于无穷，式 (64) 中对关联长度 ξ 的依赖可以替换为对时空格点单位下线度 $N^{1/4}$ 的依赖，由此得到

$$\lambda(N) = \lambda^c + \text{ const. } N^{\beta'(\lambda^c)/4}. \tag{66}$$

As we saw above, the absence of a UV fixed point could be deduced by the absence of a divergent correlation length along a trajectory of constant physics in the (μ, λ) - plane. In the finite-size scaling scenario, this is restated as N not going to infinity along any such curve of constant physics. In the case of a ϕ^4 theory in four-dimensional Spacetime, this is exactly what happens, and the conclusion is that there is no UV fixed point in the theory [59, 60] . The second-order transition line of the theory is related to the IR limit of the theory where $\lambda_R = 0$.

正如我们上文所见，若在常数物理轨迹的 (μ, λ) 平面中不存在发散的关联长度，就可以推断没有紫外固定点。在有限尺寸标度情景下，这可以重新表述为: N 在任意这类常数物理曲线上都不会趋于无穷。对于四维时空的 ϕ^4 理论，实际情况正是如此，因此结论是理论 [59, 60] 中不存在紫外固定点。该理论的二级相变线对应理论的红外极限，在此极限下 $\lambda_R = 0$ 。

Finite-Size Scaling Analysis in CDT

CDT 中的有限尺寸标度分析

We now want to apply the above formalism to CDT [33, 40] and in addition take advantage of the time-slice structure present in CDT. In fact, one does precisely that in Monte Carlo simulations on an ordinary hypercubic lattice. Instead of the point-point correlator $\langle \phi(n)\, \phi(n') \rangle$ used in (61), one averages the positions n and n' over positions $n(t_k)$ belonging to a hyperplane located at "time" $t_k = ka$ and positions $n'\left(t'_{k'}\right)$ located at "time" $t' = k'a$, where the two hyperplanes are separated by a lattice distance $d = |k - k'|$. This reduces the fluctuations of the measured correlator and it also has the advantage that power corrections to the exponential falloff of the correlator are absent, making it easier to determine the correlation length (see again [61] for details). Thus, one replaces (61) by

我们现在希望将上述形式体系应用于 CDT [33, 40]，并额外利用 CDT 中存在的时间切片结构。实际上，在普通超立方晶格上的蒙特卡洛模拟中正是这么做的。相较于式 (61) 中使用的点-点关联函数 $\langle \phi(n)\phi(n')\rangle$，我们对位置 n 和 n' 做平均，其中位置 $n(t_k)$ 属于“时间” $t_k = ka$ 处的超平面，位置 $n'\left(t'_{k'}\right)$ 属于“时间” $t' = k'a$ 处的超平面，两个超平面相隔晶格距离 $d = |k-k'|$。这可以降低测量得到的关联函数的涨落，同时它还有一个优势: 关联函数指数衰减的幂次修正不存在，使得我们更容易确定关联长度 (详见文献 [61])。因此，我们用下式替换 (61):

$$\left\langle \sum_{n(t_k)} \phi(n(t_k)) \sum_{n'(t_{k'})} \phi\left(n'\left(t'_{k'}\right)\right) \right\rangle = \text{ const. } \mathrm{e}^{-|k-k'|/\xi}. \tag{67}$$

In our CDT theory of pure geometry, we do not have a field $\phi(n)$ at our disposal, but as in the two-dimensional case we can use the ”unit” field $1(n)$ which assigns the value 1 to each four-simplex. Each three-simplex T^3 in the three-dimensional triangulation T_k^3 corresponding to time t_k belongs to two four-simplices $T^{(4,1)}\left(T^3\right)$ and $T^{(1,4)}\left(T^3\right)$. We can then write

在我们的纯几何 CDT 理论中，我们没有可用的场 $\phi(n)$，但和二维情况一样，我们可以使用“单位场” $1(n)$，它给每个四单纯形赋值 1。对应时间 t_k 的三维三角剖分 T_k^3 中的每个三单纯形 T^3 属于两个四单纯形 $T^{(4,1)}\left(T^3\right)$ 和 $T^{(1,4)}\left(T^3\right)$。于是我们可以写出

$$\sum_{n(t_k)} \phi(n(t_k)) \to \frac{1}{2} \sum_{T^3 \in T_k^3} \left(1\left(T^{(4,1)}\left(T^3\right)\right) + 1\left(T^{(1,4)}\left(T^3\right)\right)\right) = N_3(t_k), \tag{68}$$

where $N_3(t_k)$ is the number of three-simplices in the spatial slice at time t_k. Since $\langle N_3(t_k)\rangle > 0$, we have a situation like in the broken phase of a ϕ^4 theory: connected correlators have to be expanded around $\langle\phi\rangle \neq 0$. Here one has to expand around $\langle N_3(t_k)\rangle > 0$ (see Eq. (71) below). Moreover, it turns out that $\langle N_3(t_k)\rangle$ will be a highly nontrivial function of t_k; see Fig. 4, once the translational invariance of the action in t is dealt with in the proper way. In the symmetric phase of the ϕ^4 theory, all correlators with an odd number of fields will be zero, and the two- and the four-point correlators are needed in order to study the renormalization flow of μ and λ. Here, in CDT, we can extract information of the flow of k_0 and Δ by considering only one- and two-point correlators because the one-point function $\langle N_3(t_k)\rangle$ has a nontrivial dependence on t_k (The nontrivial dependence on t_k is shown in Fig. 4, and it only appears after the zero mode corresponding to translational invariance in t has been eliminated. The condition $H(0) = 1$ then refers to the time $t_0 = 0$ that is chosen as the maximum of the ”blob” $\langle N_3(t_k)\rangle$.)

其中 $N_3(t_k)$ 是时间 t_k 处空间切片中三单纯形的数量。由于 $\langle N_3(t_k)\rangle > 0$，我们得到的情况和 ϕ^4 理论的破缺相类似: 连通关联函数需要围绕 $\langle\phi\rangle \neq 0$ 展开。此处我们需要围绕 $\langle N_3(t_k)\rangle > 0$ 展开 (见下文式 (71))。此外，一旦适当处理了作用量在 t 上的平移不变性，结果表明 $\langle N_3(t_k)\rangle$ 会是 t_k 的高度非平凡函数；参见图 4。在 ϕ^4 理论的对称相中，所有含奇数个场的关联函数都为零，要研究 μ 和 λ 的重整化流需要用到两点和四点关联函数。而在 CDT 中，由于一点函数 $\langle N_3(t_k)\rangle$ 对 t_k 存在非平凡依赖，我们仅通过考虑一点和两点关联函数就可以提取 k_0 和 Δ 的流的信息 ($\langle N_3(t_k)\rangle$ 对 t_k 的非平凡依赖如图 4 所示，它仅在消除对应 t 平移不变性的零模后才会出现。条件 $H(0) = 1$ 对应被选作“团” $\langle N_3(t_k)\rangle$ 最大值的时间 $t_0 = 0$。)

$$\langle N_3(t_k)\rangle_{N_4} \propto N_4^{3/4} H\left(\frac{k}{N_4^{1/4}}\right), \; H(0) = 1 \tag{69}$$

$$\langle N_3(t_k) N_3(t'_{k'}) \rangle^c_{N_4} \propto N_4 F\left(\frac{k}{N_4^{1/4}}, \frac{k'}{N_4^{1/4}}\right), \tag{70}$$

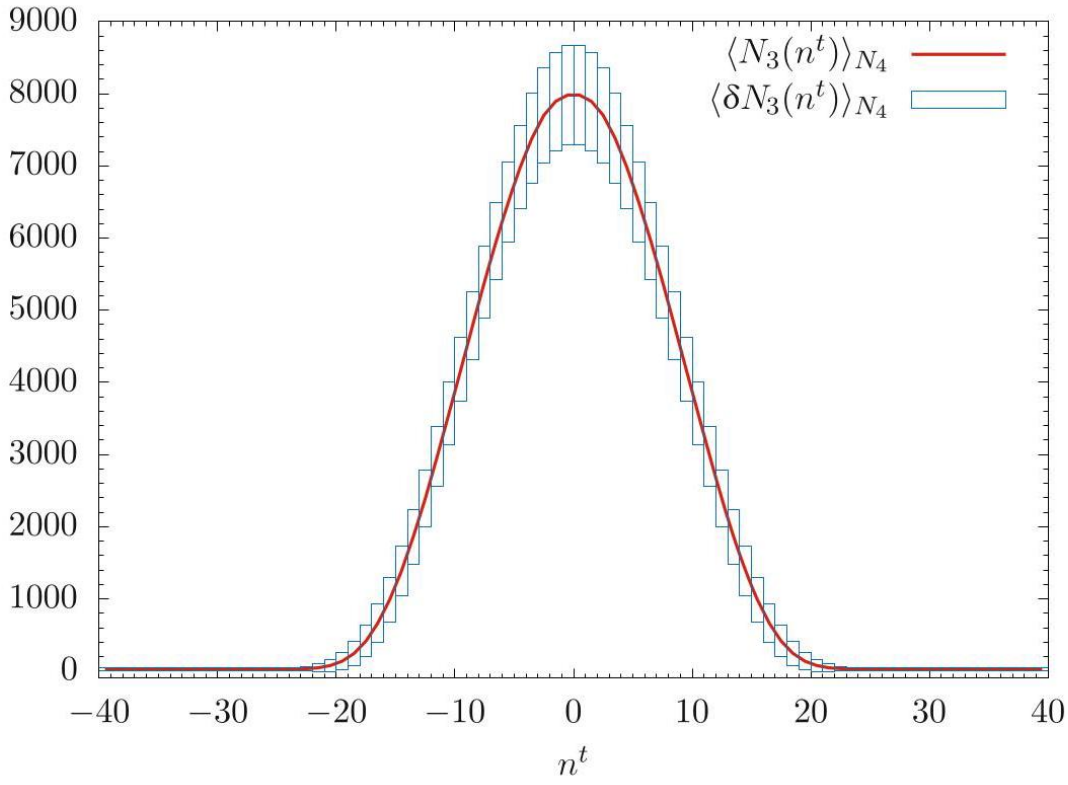

Fig. 4 The average spatial volume $\langle N_3(t_k) \rangle_{N_4}$ as a result of MC measurements for $N_4 = 362.000$. The best fit of the form (73) yields a curve which cannot be distinguished from the measured average at the given plot resolution. The bars indicate the average size of quantum fluctuations $\delta N_3(t_k)$

图 4 蒙特卡洛测量得到的平均空间体积 $\langle N_3(t_k) \rangle_{N_4}$ ，对应 $N_4 = 362.000$ 。形式为 (73) 的最佳拟合得到的曲线在图中现有分辨率下与测量平均值无法区分。误差棒表示量子涨落 $\delta N_3(t_k)$ 的平均大小

where the connected correlator, as in (61), is defined by

其中连通关联函数如 (61) 所示，定义为

$$\langle N_3(t_k) N_3(t'_{k'}) \rangle^c_{N_4} = \langle (N_3(t_k) - \langle N_3(t_k) \rangle)((N_3(t'_{k'}) - \langle N_3(t'_{k'}) \rangle)) \rangle_{N_4}. \tag{71}$$

In particular we have

我们可得

$$\langle \delta N_3(t_k) \rangle_{N_4} := \sqrt{\langle N_3^2(t_k) \rangle^c_{N_4}} \propto \sqrt{N_4} F\left(\frac{k}{N_4^{1/4}}, \frac{k}{N_4^{1/4}}\right), \; F(0,0) = 1. \tag{72}$$

Monte Carlo simulations confirm the functional form alluded to in (69) and (70) when we are in phase C_{dS} . In particular we have measured with high precision [24, 25]

当处于相 C_{dS} 时，蒙特卡洛模拟证实了 (69) 和 (70) 中提到的函数形式。我们已经高精度测量得到 [24, 25]

$$H\left(\frac{k}{N_4^{1/4}}\right) \propto \cos^3\left(\frac{k}{\omega N_4^{1/4}}\right) \tag{73}$$

where ω depends on k_0 and Δ . This functional form is the reason why we call phase C_{dS} the ”de Sitter” phase. It is the functional form that $N_3(t_k)$ would have for a four-sphere where t_k denotes the geodesic distance from the three-equator. It is valid for

其中 ω 依赖于 k_0 和 Δ 。该函数形式就是我们将 C_{dS} 相称为“德西特”相的原因。对于四球来说，$N_3(t_k)$ 满足该函数形式，其中 t_k 表示到三维赤道面的测地线距离。它适用于

$$-\frac{\pi}{2}\omega N_4^{1/4} < k < \frac{\pi}{2}\omega N_4^{1/4}, \tag{74}$$

where we for convenience have chosen to locate the maximum of $\langle N_3(t_k)\rangle_{N_4}$ at $k = 0$. In the computer simulations which resulted in this distribution, the lattice time extension was chosen larger than $\pi\omega N_4^{1/4}$. Outside the region (74) one finds $N_3(t_k) \approx 0$ which is of the order of the cutoff, i.e., the smallest S^3 one can create by gluing together 5 tetrahedra. This is why it does not matter whether we choose the time direction to correspond to S^1 or to $[0, 1]$. Figure 4 shows the measured three-volume profile (69), as well as the theoretical curve (73), and finally the fluctuations (72) around the measured three-volume profile.

其中为了方便，我们选择将 $\langle N_3(t_k)\rangle_{N_4}$ 的最大值定位在 $k = 0$ 处。在得到该分布的计算机模拟中，格点时间延拓被设置为大于 $\pi\omega N_4^{1/4}$ 。在区域 (74) 外，可得到 $N_3(t_k) \approx 0$ ，其大小约为截断尺度，即拼接 5 个四面体能得到的最小 S^3 。因此无论我们选择时间方向对应 S^1 还是 $[0, 1]$ 都不影响结果。图 4 展示了测得的三维体积轮廓 (69)、理论曲线 (73)，以及测得三维体积轮廓周围的涨落 (72)。

Equations (69) and (72) allow us in principle to follow a path of constant continuum physics in the (k_0, Δ) coupling-constant space, which might lead to a UV fixed point, in the spirit of the finite-size scaling discussion for the ϕ^4 theory. We define the continuum three-volume of a time slice as

原则上，方程 (69) 和 (72) 允许我们在 (k_0, Δ) 耦合常数空间中追踪恒定连续物理的路径，按照有限尺寸标度讨论的思路，这可能会得到 ϕ^4 理论的紫外不动点。我们将时间切片的连续三维体积定义为

$$V_3(t_k) = \frac{\sqrt{2}}{12}a^3 N_3(t_k), \; t_k = ka. \tag{75}$$

First, for fixed (k_0, Δ) , taking $N_4 \to \infty$, we see that

首先，对于固定的 (k_0, Δ) ，取 $N_4 \to \infty$ ，可得

$$\left.\frac{\delta V_3(t_k)}{V_3(t_k)}\right|_{N_4} = \left.\frac{\langle \delta N_3(t_k)\rangle}{\langle N_3(t_k)\rangle}\right|_{N_4} \propto \frac{1}{N_4^{1/4}} \to 0 \;\text{ for }\; N_4 \to \infty. \tag{76}$$

The simplest interpretation of this result is that for fixed (k_0, Δ) we should view the lattice spacing a in (75) as constant. Then $N_4 \to \infty$ implies that $V_3 \to \infty$ and for large continuum $V_3(t)$ one expects that the fluctuations will be small relative to $V_3(t)$. However, in the spirit of finite-size scaling, we are interested in a limit where $V_3(t)$ stays finite when $N_4 \to \infty$. This is clearly a limit where also the fluctuations around $V_3(t)$ will stay finite, since they then represent the "real" continuum fluctuations around $V_3(t)$, and they should be independent of N_4 for sufficient large N_4 . We therefore require

该结果最简单的解释是: 对于固定 (k_0, Δ) ， 我们应认为 (75) 中的格点间距 a 是常数。那么 $N_4 \to \infty$ 意味着 $V_3 \to \infty$ ，对于大的连续 $V_3(t)$ ，可以预期涨落相对于 $V_3(t)$ 会很小。但按照有限尺寸标度的思路，我们感兴趣的是当 $N_4 \to \infty$ 时 $V_3(t)$ 仍保持有限的极限。显然在该极限下，$V_3(t)$ 周围的涨落也会保持有限，因为它们代表“真实”的连续涨落，当 N_4 足够大时，涨落应与 N_4 无关。因此我们要求

$$\frac{\delta V_3(t)}{V_3(t)} = \frac{\langle \delta N_3(t) \rangle}{\langle N_3(t) \rangle} = \text{ const. for fixed } t = ka \propto k N_4^{-1/4}. \tag{77}$$

According to (76), we can only obtain this by changing (k_0, Δ) when we change N_4 . We thus have precisely the picture advocated in the ϕ^4 case: the requirement of constant continuum physics leads to a path in the bare, dimensionless lattice coupling-constant space when we increase N_4 . If this path continues to $N_4 \to \infty$, then according to (75) the lattice spacing $a \to 0$ and we might reach a UV fixed point (k_0^c, Δ^c) .

根据式 (76)，我们只有在改变 N_4 时通过改变 (k_0, Δ) 才能得到这个结果。由此我们得到了和 ϕ^4 情况完全一致的图景: 当我们增大 N_4 时，保持连续统物理性质不变的要求会在裸无量纲晶格耦合常数空间中给出一条路径。如果这条路径延伸至 $N_4 \to \infty$ ，根据式 (75)，晶格间距 $a \to 0$ ，我们就有可能到达一个紫外固定点 (k_0^c, Δ^c) 。

In [33] it was attempted to follow this program, but it was before the $C_b - C_{dS}$ phase transition line was discovered and the N_4 used might have been too small, so the question about the existence of a UV fixed point associated with the $C_b - C_{dS}$ phase transition line is still open.

文献 [33] 曾尝试推进这一研究，但该工作完成于 $C_b - C_{dS}$ 相变线被发现之前，且所用的 N_4 可能偏小，因此与 $C_b - C_{dS}$ 相变线相关的紫外固定点是否存在这一问题目前仍无定论。

A different approach, in some sense closer to the renormalization group approach, is to use the Monte Carlo simulation data to construct an effective action for the three-volume $N_3(t_k)$. The corresponding effective action can be viewed as a kind of minisuperspace action. However, the action is (numerically) derived from the full quantum theory, so the geometric degrees of freedom different from $N_3(t)$ have not been ignored, but rather (numerically) integrated out. - Chapter 82, "Semiclassical and Continuum Limits of Four-Dimensional CDT" in this section of the handbook will discuss this in detail. So far, it has not been possible to identify in an unambiguous way a UV fixed point, but the formalism for doing so now exists, as explained above, and future computer simulations can hopefully clarify if the fixed point exists or not.

还有一种不同的研究方法，在某种意义上更接近重整化群方法，它利用蒙特卡洛模拟数据构建三体积 $N_3(t_k)$ 的有效作用量。对应的有效作用量可以看作一种微超空间作用量。不过该作用量是从全量子理论 (数值) 导出的，因此不同于 $N_3(t)$ 的几何自由度并未被忽略，而是已经 (数值) 积分掉了——本手册本章的第 82 章《四维 CDT 的半经典极限与连续统极限》会详细讨论这一内容。到目前为止，人们还无法明确识别出紫外固定点，但正如上文所述，相关研究框架已经建立，未来的计算机模拟有望厘清该固定点是否存在。

Future Perspectives

未来展望

In discussing four-dimensional CDT we have focused on how one can in principle locate a UV fixed point. If it exists and if it is nontrivial, it would be a strong indication that there exists a non-perturbative, unitary quantum field theory of Lorentzian geometries at all length scales. It could be the quantum theory of GR. Of course one would have to provide convincing arguments in favor of such an interpretation. The CDT theory is most likely unitary when rotated back from Euclidean spacetime to Lorentzian spacetime, since one can show that the Euclidean rotated version of CDT is reflection positive, a property that usually ensures that when one rotates back to Lorentzian signature, one obtains a unitary theory. We expect a quantum theory of GR to be unitary, so a putative continuum 4d CDT theory passes that test. It also makes it unlikely that the continuum limit of 4 d CDT should be some generic R^2 version of GR, since these theories typically will be non-unitary theories. Another test is that classical GR should emerge from the quantum effective action in the limit where $\hbar \to 0$. Such a test could be performed if one could construct the effective action of the quantum theory. As mentioned the effective action has been constructed for the three-volume $V_3(t)$ and it is closely related to a GR minisuperspace action. However, the real test would be to construct the full quantum effective action from the MC data and take the $\hbar \to 0$ limit, and we do not yet know how to do that. Maybe the simplest way to relate the lattice theory associated with a UV fixed point to continuum gravity theories is to determine the critical exponents related to the lattice fixed point and compare with similar analytic renormalization group calculations. In principle the functional renormalization group approach should provide us with unique critical exponents if a UV quantum gravity fixed point exists. In practice the calculations in both approaches will have error bars and the comparison might not be easy.

在讨论四维因果动态三角剖分 (CDT) 时，我们重点讨论了原则上如何定位紫外不动点。如果该不动点存在且是非平凡的，将有力表明在所有尺度上都存在一个关于洛伦兹几何的非微扰幺正量子场论。它可能就是广义相对论的量子理论。当然，我们需要提供有说服力的论据来支持这一解释。将 CDT 从欧几里得时空旋转回洛伦兹时空后，CDT 理论极大概率是幺正的，因为可以证明，经过欧几里得旋转的 CDT 满足反射正性——这一性质通常可以保证旋转回洛伦兹号差后得到的是幺正理论。我们预期广义相对论的量子理论是幺正的，因此假想的连续四维 CDT 理论通过了这一检验。这也说明 4 d CDT 的连续极限不太可能是广义相对论的某种通用 R^2 版本，因为这类理论通常都是非幺正的。另一项检验是，经典广义相对论应当能在 $\hbar \to 0$ 的极限下从量子有效作用量中涌现出来。如果我们能构造出量子理论的有效作用量，就可以开展这项检验。如前文所述，三维体积的有效作用量 $V_3(t)$ 已经构造完成，且它与广义相对论的超空间 minisuperspace 作用量密切相关。不过真正的检验是从蒙特卡洛数据构造完整的量子有效作用量，再取 $\hbar \to 0$ 极限，而我们目前还不知道该如何做到这一点。若要将紫外不动点对应的格点理论与连续引力理论联系起来，最简单的方法或许是确定格点不动点相关的临界指数，再与解析重整化群计算得到的类似结果对比。原则上，如果量子引力的紫外不动点存在，泛函重整化群方法应该能为我们给出唯一的临界指数。实际上，两种方法的计算都存在误差，比较起来可能并不容易。

There is a number of conceptional issues in a theory of quantum gravity. We tried to illustrate these in the case of solvable two-dimensional gravity models. How do we talk about distances in a theory of quantum gravity where in the path integral we integrate over the geometries that determine distances? Does it make sense to talk about arbitrarily small distances, much smaller than the quantum fluctuations of geometries? The solvable models of two-dimensional gravity encourage us to believe that it can make sense and that we are not forced to endorse the common statement that ”the concept of geometry has to break down at short distances.” Of course the situation could be different in four dimensions and there is little hope that we can solve the four-dimensional theories analytically. However, the Monte Carlo simulations of the four-dimensional quantum gravity models might teach us how we should think about geometry at the shortest distances. In the two-dimensional models there have been very fruitful interplays between pure theory and Monte Carlo simulations of the models. What can relatively easily be measured in the Monte Carlo simulations are the fractal dimensions of the spacetime geometries, both the so-called Hausdorff dimension and spectral dimension (This is discussed in Chap. 78, ”Spectral Observables and Gauge Field Couplings in Causal Dynamical Triangulations” in this section of the handbook.). Let us just mention that the measurement of the spectral dimension in 4d CDT resulted in the surprising result that the dimension seems to be scale-dependent [22]. Inspired by this, similar results have been obtained analytically in a number of quantum gravity models. This is an example of a fruitful interplay between numerical studies and analytic calculations also in higher than two dimensions.

量子引力理论存在许多概念性问题。我们已经尝试以可解的二维引力模型为例说明这些问题。在路径积分中，我们对决定距离的几何做积分，那在这样的量子引力理论中我们该如何描述距离？谈论远小于几何量子涨落尺度的任意小距离有意义吗？二维可解引力模型让我们有理由相信这是有意义的，我们不必被迫接受那个常见论断——“几何的概念必然在短距离失效”。当然，四维的情况可能不同，我们几乎没有希望解析求解四维理论。不过，四维量子引力模型的蒙特卡洛模拟或许能教会我们，应该如何看待极短距离下的几何。在二维模型中，纯理论研究与模型的蒙特卡洛模拟之间一直存在卓有成效的相互影响。在蒙特卡洛模拟中，比较容易测量的是时空几何的分形维数，包括所谓的豪斯多夫维数和谱维数 (本手册本册的第 78 章“因果动态三角剖分中的谱可观测量与规范场耦合”对此已有讨论)。我们只需要指出，四维 CDT 中的谱维数测量得到了一个惊人结果: 维度似乎是依赖于尺度的 [22]。受此启发，许多量子引力模型中都已经解析得到了类似结果。这就是高于二维的情况下，数值研究与解析计算之间富有成效的相互影响的一个例子。

Presently we do not know for certain if there exists a UV fixed point that will allow us to define an ”ordinary” quantum field theory of quantum gravity at all scales. However, the lattice efforts will not be wasted even if it should turn out that such a fixed point does not exist. First of all there exists most likely in ordinary continuum gravity an effective quantum field theory up to energies of the order of the Planck energy. There are no conceptual problems also using such an effective theory in a cosmological context. It will be a quantum field theory with a cutoff. The lattice theories, both four-dimensional EDT and CDT, will be such theories, where the lattice spacing acts as the cutoff. Such cutoff theories can still provide us with a lot of non-perturbative information about (the effective theory of) quantum gravity, since non-perturbative information is not necessarily linked to short-distance phenomena. In fact, an example of this is provided by one of the first cases where non-perturbative topological aspects entered into quantum field theory, namely, in the three-dimensional Georgi-Glashow model. This model is invariant under local $SO(3)$ gauge transformations. It has a non-Abelian gauge field and a three-component scalar field that transforms as a vector under the action of the $SO(3)$ gauge group. When rotated to Euclidean spacetime the model has classical monopole solutions that act as instantons. The Higgs potential is such that the $SO(3)$ symmetry is broken down to a $U(1)$ symmetry with an associated Abelian gauge field, while the two other gauge field components combine to a massive charged, so-called W particle. The monopoles will result in confinement of particles with $U(1)$ charge [64]. Amazingly, a pure $U(1)$ lattice gauge theory will produce exactly the same non-perturbative physics because it has lattice monopoles [63]. These monopoles have a mass which diverges as $1/a$, the lattice spacing, while the mass of the monopoles in the Georgi-Glashow model is proportional to m_w/e^2 , where m_w is the mass of the W particle and e the charge associated to the $U(1)$ symmetry. The lattice physics is therefore identical to the physics of the Georgi-Glashow model for distances larger than the lattice spacing a , as long as $a \sim e^2/m_w$. However, we cannot take $a < e^2/m_w$ and still capture the physics of the Georgi-Glashow models. When $a \to 0$, the lattice monopoles will be infinitely heavy and decouple, and we will just obtain a theory with a free photon. If we have a situation in quantum gravity, where there is no UV fixed point, there might be new degrees of freedom for distances shorter than the Planck length, and we will not be able to represent them correctly when our lattice spacing a is less than the Planck length. But physics at scales larger than the Planck length, including some non-perturbative physics caused by these unknown degrees of freedom, could still be correctly described by our lattice models. In the case of CDT we have, as mentioned above, constructed an effective minisuperspace action, which may describe physics well all the way down to the Planck length and allow us to study universes which are not much larger than the Planck length and discover corrections to the simplest minisuperspace models. In addition, we can also study the non-perturbative interaction between matter fields and gravity in a full quantum context (The nontrivial interaction between geometry and matter is described in Chap. 79, ”Scalar Fields in Four-Dimensional CDT”, of this section of the handbook). But hopefully there is a UV fixed point. Then such studies will not be confined to distances larger than the Planck length.

目前我们尚不确定是否存在一个紫外不动点，能让我们在所有尺度上定义一套“普通”的量子引力量子场论。但即便最终证明这类不动点并不存在，格点研究的努力也不会白费。首先，在普通连续引力中，极有可能存在一个适用至普朗克能标量级的有效量子场论。将这类有效理论应用于宇宙学背景也不存在概念性问题，它本身就是一个带截断的量子场论。四维 EDT 和 CDT 这两类格点理论都属于这类带截断的理论，格点间距在这里就充当截断。这类带截断的理论仍然可以为我们提供大量关于 (量子引力有效理论的) 非微扰信息，因为非微扰信息并不一定和短距现象绑定。实际上，非微扰拓扑性质进入量子场论的最早案例之一就能说明这一点，也就是三维乔治-格拉肖模型。该模型在局域 $SO(3)$ 规范变换下不变，它包含一个非阿贝尔规范场和一个三分量标量场，后者在 $SO(3)$ 规范群作用下按矢量变换。将其旋转到欧几里得时空后，该模型存在作为瞬子的经典磁单极解。希格斯势会将 $SO(3)$ 对称性破缺为 $U(1)$ 对称性，伴随产生一个阿贝尔规范场，另外两个规范场分量则结合为带质量的带电粒子，即所谓的 W 粒子。磁单极会导致带 $U(1)$ 电荷的粒子被禁闭 [64]。令人惊讶的是，纯 $U(1)$ 格点规范论能得到完全一致的非微扰物理，因为它本身就存在格点磁单极 [63]。这些格点磁单极的质量会随格点间距 $1/a$ 发散，而乔治-格拉肖模型中磁单极的质量正比于 m_w/e^2 ，其中 m_w 是 W 粒子的质量，e 是 $U(1)$ 对称性对应的电荷。因此对于远大于格点间距 a 的尺度，只要满足 $a \sim e^2/m_w$ ，格点物理就和乔治-格拉肖模型的物理完全一致。但我们不可能在让 $a < e^2/m_w$ 趋于零的同时仍然保留乔治-格拉肖模型的物理。当 $a \to 0$ 时，格点磁单极会变得无限重并退耦合，我们最终只会得到一个自由光子的理论。如果量子引力中确实不存在紫外不动点，那么短于普朗克长度的尺度可能会存在新的自由度，当我们的格点间距 a 小于普朗克长度时，我们无法正确表示这些自由度。但大于普朗克长度尺度的物理，包括这些未知自由度引发的部分非微扰物理，仍然可以被我们的格点模型正确描述。对于 CDT，如上文中提到的，我们已经构造了一个有效超空间极小作用量，它可以很好地描述一直到普朗克长度的物理，允许我们研究尺寸不比普朗克长度大很多的宇宙，并得到最简单超空间极小模型的修正项。此外，我们还可以在完整量子背景下研究物质场与引力的非微扰相互作用 (几何与物质的非平凡相互作用已在本手册本章的第 79 章“四维 CDT 中的标量场”中介绍)。但我们依然希望存在紫外不动点，若真如此，这类研究就不必局限于大于普朗克长度的尺度了。

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